1.3
Predicates and Quantifiers

INTRODUCTION

Statements involving variables, such as

\[ x > 3, \quad x = y + 3, \quad \text{and} \quad x + y = z, \]

are often found in mathematical assertions and in computer programs. These statements are neither true nor false when the values of the variables are not specified. In this section we will discuss the ways that propositions can be produced from such statements.

The statement “\( x \) is greater than 3” has two parts. The first part, the variable \( x \), is the subject of the statement. The second part—the predicate, “is greater than 3”—refers to a property that the subject of the statement can have. We can denote the statement “\( x \) is greater than 3” by \( P(x) \), where \( P \) denotes the predicate “is greater than 3” and \( x \) is the variable. The statement \( P(x) \) is also said to be the value of the propositional function \( P \) at \( x \). Once a value has been assigned to the variable \( x \), the statement \( P(x) \) becomes a proposition and has a truth value. Consider the following example.

**Example 1**

Let \( P(x) \) denote the statement “\( x \) > 3.” What are the truth values of \( P(4) \) and \( P(2) \)?

Charles Sanders Peirce (1839–1914). Many consider Charles Peirce the most original and versatile intellect from the United States; he was born in Cambridge, Massachusetts. His father, Benjamin Peirce, was a professor of mathematics and natural philosophy at Harvard. Peirce attended Harvard (1855–1859) and received a Harvard master of arts degree (1862) and an advanced degree in chemistry from the Lawrence Scientific School (1863). His father encouraged him to pursue a career in science, but instead he chose to study logic and scientific methodology.

In 1861, Peirce became an aide in the United States Coast Survey, with the goal of better understanding scientific methodology. His service for the Survey exempted him from military service during the Civil War. While working for the Survey, Peirce carried out astronomical and geodetic work. He made fundamental contributions to the design of pendulums and to map projections, applying new mathematical developments in the theory of elliptic functions. He was the first person to use the wavelength of light as a unit of measurement. Peirce rose to the position of Assistant for the Survey, a position he held until he was forced to resign in 1891 when he disagreed with the direction taken by the Survey's new administration.

Although making his living from work in the physical sciences, Peirce developed a hierarchy of sciences, with mathematics at the top rung, in which the methods of one science could be adapted for use by those sciences under it in the hierarchy. He was also the founder of the American philosophical theory of pragmatism.

The only academic position Peirce ever held was as a lecturer in logic at Johns Hopkins University in Baltimore from 1879 to 1884. His mathematical work during this time included contributions to logic, set theory, abstract algebra, and the philosophy of mathematics. His work is still relevant today; some of his work on logic has been recently applied to artificial intelligence. Peirce believed that the study of mathematics could develop the mind's powers of imagination, abstraction, and generalization. His diverse activities after retiring from the Survey included writing for newspapers and journals, contributing to scholarly dictionaries, translating scientific papers, guest lecturing, and textbook writing. Unfortunately, the income from these pursuits was insufficient to protect him and his second wife from abject poverty. He was supported in his later years by a fund created by his many admirers and administered by the philosopher William James, his lifelong friend. Although Peirce wrote and published voluminously in a vast range of subjects, he left more than 100,000 pages of unpublished manuscripts. Because of the difficulty of studying his unpublished writings, scholars have only recently started to understand some of his varied contributions. A group of people is devoted to making his work available over the Internet to bring a better appreciation of Peirce's accomplishments to the world.
Solution: The statement $P(4)$ is obtained by setting $x = 4$ in the statement “$x > 3$.” Hence, $P(4)$, which is the statement “$4 > 3$,” is true. However, $P(2)$, which is the statement “$2 > 3$,” is false.

We can also have statements that involve more than one variable. For instance, consider the statement “$x = y + 3$.” We can denote this statement by $Q(x, y)$, where $x$ and $y$ are variables and $Q$ is the predicate. When values are assigned to the variables $x$ and $y$, the statement $Q(x, y)$ has a truth value.

**Example 2**

Let $Q(x, y)$ denote the statement “$x = y + 3$.” What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

**Solution:** To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement “$1 = 2 + 3$,” which is false. The statement $Q(3, 0)$ is the proposition “$3 = 0 + 3$,” which is true.

Similarly, we can let $R(x, y, z)$ denote the statement “$x + y = z$.” When values are assigned to the variables $x$, $y$, and $z$, this statement has a truth value.

**Example 3**

What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

**Solution:** The proposition $R(1, 2, 3)$ is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement “$1 + 2 = 3$,” which is true. Also note that $R(0, 0, 1)$, which is the statement “$0 + 0 = 1$,” is false.

In general, a statement involving the $n$ variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function $P$ at the $n$-tuple $(x_1, x_2, \ldots, x_n)$, and $P$ is also called a predicate.

Propositional functions occur in computer programs, as the following example demonstrates.

**Example 4**

Consider the statement

\[ \text{if } x > 0 \text{ then } x := x + 1. \]

When this statement is encountered in a program, the value of the variable $x$ at that point in the execution of the program is inserted into $P(x)$, which is “$x > 0$.” If $P(x)$ is true for this value of $x$, the assignment statement $x := x + 1$ is executed, so the value of $x$ is increased by 1. If $P(x)$ is false for this value of $x$, the assignment statement is not executed, so the value of $x$ is not changed.
QUANTIFIERS

When all the variables in a propositional function are assigned values, the resulting statement has a truth value. However, there is another important way, called quantification, to create a proposition from a propositional function. Two types of quantification will be discussed here, namely, universal quantification and existential quantification.

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the universe of discourse. Such a statement is expressed using a universal quantification. The universal quantification of a propositional function is the proposition that asserts that \( P(x) \) is true for all values of \( x \) in the universe of discourse. The universe of discourse specifies the possible values of the variable \( x \).

**DEFINITION 1.** The universal quantification of \( P(x) \) is the proposition

\[
(P(x) \text{ is true for all values of } x \text{ in the universe of discourse}.
\]

The notation

\[ \forall x P(x) \]

denotes the universal quantification of \( P(x) \). Here \( \forall \) is called the universal quantifier. The proposition \( \forall x P(x) \) is also expressed as

"for all \( x \) \( P(x) \)" or "for every \( x \) \( P(x) \)."

**Remark:** It is best to avoid the word "any" since it is often ambiguous as to whether it means "every" or "some." In some cases, "any" is unambiguous, such as when it is used in negatives, for example, "there is not any reason not to study hard."

**EXAMPLE 5**

Express the statement

"Every student in this class has studied calculus"

as a universal quantification.

**Solution:** Let \( P(x) \) denote the statement

"\( x \) has studied calculus."

Then the statement "Every student in this class has studied calculus" can be written as

\[ \forall x P(x) \]

where the universe of discourse consists of the students in this class. This statement can also be expressed as

\[ \forall x (S(x) \rightarrow P(x)) \]

where \( S(x) \) is the statement

"\( x \) is in this class."

\( P(x) \) is as before, and the universe of discourse is the set of all students.

**Example 5** illustrates that there is often more than one good way to express a quantification.
EXAMPLE 6

Let \( P(x) \) be the statement \( \langle x + 1 > x \rangle \). What is the truth value of the quantification \( \forall x \ P(x) \), where the universe of discourse is the set of real numbers?

**Solution:** Since \( P(x) \) is true for all real numbers \( x \), the quantification

\[ \forall x \ P(x) \]

is true.

EXAMPLE 7

Let \( Q(x) \) be the statement \( \langle x < 2 \rangle \). What is the truth value of the quantification \( \forall x \ Q(x) \), where the universe of discourse is the set of real numbers?

**Solution:** \( Q(x) \) is not true for all real numbers \( x \), since, for instance, \( Q(3) \) is false. Thus

\[ \forall x \ Q(x) \]

is false.

When all of the elements in the universe of discourse can be listed—say, \( x_1, x_2, \ldots, x_n \)—it follows that the universal quantification \( \forall x \ P(x) \) is the same as the conjunction

\[ P(x_1) \land P(x_2) \land \cdots \land P(x_n), \]

since this conjunction is true if and only if \( P(x_1), P(x_2), \ldots, P(x_n) \) are all true.

EXAMPLE 8

What is the truth value of \( \forall x \ P(x) \), where \( P(x) \) is the statement \( \langle x^2 < 10 \rangle \) and the universe of discourse consists of the positive integers not exceeding 4?

**Solution:** The statement \( \forall x \ P(x) \) is the same as the conjunction

\[ P(1) \land P(2) \land P(3) \land P(4), \]

since the universe of discourse consists of the integers 1, 2, 3, and 4. Since \( P(4) \), which is the statement \( \langle 4^2 < 10 \rangle \), is false, it follows that \( \forall x \ P(x) \) is false.

Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification. With existential quantification, we form a proposition that is true if and only if \( P(x) \) is true for at least one value of \( x \) in the universe of discourse.

**DEFINITION 2.** The existential quantification of \( P(x) \) is the proposition

"There exists an element \( x \) in the universe of discourse such that \( P(x) \) is true."

We use the notation

\[ \exists x \ P(x) \]

for the existential quantification of \( P(x) \). Here \( \exists \) is called the existential quantifier. The existential quantification \( \exists x \ P(x) \) is also expressed as

"There is an \( x \) such that \( P(x) \),"

"There is at least one \( x \) such that \( P(x) \),"
or

“For some $x \ P(x),$$

**EXAMPLE 9**

Let $P(x)$ denote the statement “$x > 3$.” What is the truth value of the quantification $\exists x \ P(x)$, where the universe of discourse is the set of real numbers?

**Solution:** Since “$x > 3$” is true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x \ P(x)$, is true.

**EXAMPLE 10**

Let $Q(x)$ denote the statement “$x = x + 1$.” What is the truth value of the quantification $\exists x \ Q(x)$, where the universe of discourse is the set of real numbers?

**Solution:** Since $Q(x)$ is false for every real number $x$, the existential quantification of $Q(x)$, which is $\exists x \ Q(x)$, is false.

When all of the elements in the universe of discourse can be listed—say, $x_1, x_2, \ldots, x_n$—the existential quantification $\exists x \ P(x)$ is the same as the disjunction

$$P(x_1) \lor P(x_2) \lor \cdots \lor P(x_n),$$

since this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

**EXAMPLE 11**

What is the truth value of $\exists x \ P(x)$ where $P(x)$ is the statement “$x^2 > 10$” and the universe of discourse consists of the positive integers not exceeding 4?

**Solution:** Since the universe of discourse is $\{1, 2, 3, 4\}$, the proposition $\exists x \ P(x)$ is the same as the disjunction

$$P(1) \lor P(2) \lor P(3) \lor P(4).$$

Since $P(4)$, which is the statement “$4^2 > 10$,” is true, it follows that $\exists x \ P(x)$ is true.

Table 1 summarizes the meaning of the universal and the existential quantifiers.

It is sometimes helpful to think in terms of looping and searching when determining the truth value of a quantification. Suppose that there are $n$ objects in the universe of discourse for the variable $x$. To determine whether $\forall x \ P(x)$ is true, we can loop through all $n$ values of $x$ to see if $P(x)$ is always true. If we encounter a value $x$ for which $P(x)$ is false, then we have shown that $\forall x \ P(x)$ is false. Otherwise, $\forall x \ P(x)$ is true. To see whether $\exists x \ P(x)$ is true, we loop through the $n$ values of $x$ searching for a value for

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<thead>
<tr>
<th><strong>TABLE 1</strong> Quantifiers.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Statement</strong></td>
</tr>
<tr>
<td>$\forall x \ P(x)$</td>
</tr>
<tr>
<td>$\exists x \ P(x)$</td>
</tr>
</tbody>
</table>
which $P(x)$ is true. If we find one, then $\exists x P(x)$ is true. If we never find such an $x$, we have determined that $\exists x P(x)$ is false. (Note that this searching procedure does not apply if there are infinitely many values in the universe of discourse. However, it is still a useful way of thinking about the truth values of quantifications.)

Sometimes expressions involving quantifiers can be quite complicated. Translating a complicated expression into English helps understanding of its meaning. The first step in this translation is to write out what each quantifier means. The next step is to express this meaning in a simpler sentence. Consider the following examples.

**EXAMPLE 12**

Translate the statement

$$\forall x (C(x) \lor \exists y (C(y) \land F(x,y)))$$

into English, where $C(x)$ is "x has a computer," $F(x,y)$ is "x and y are friends," and the universe of discourse for both $x$ and $y$ is the set of all students at your school.

**Solution:** The statement says that for every student $x$ in your school $x$ has a computer or there is a student $y$ such that $y$ has a computer and $x$ and $y$ are friends. In other words, every student in your school has a computer or has a friend who has a computer.

**EXAMPLE 13**

Translate the statement

$$\exists x \forall y \forall z (((F(x,y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z)))$$

into English, where $F(a,b)$ means $a$ and $b$ are friends and the universe of discourse for $x$, $y$, and $z$ is the set of all students in your school.

**Solution:** This statement says that there is a student $x$ such that for all students $y$ and all students $z$ other than $y$, if $x$ and $y$ are friends and $x$ and $z$ are friends, then $y$ and $z$ are not friends. In other words, there is a student none of whose friends are also friends with each other.

Complicated expressions involving quantifiers also arise in mathematical statements. This is illustrated in the following example.

**EXAMPLE 14**

Assume that the universe of discourse for the variables $x$ and $y$ is the set of all real numbers. The statement

$$\forall x \forall y (x + y = y + x)$$

says that $x + y = y + x$ for all real numbers $x$ and $y$. This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number $x$ there is a real number $y$ such that $x + y = 0$. This states that every real number has an additive inverse. Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers.
TRANSLATING SENTENCES INTO LOGICAL EXPRESSIONS

In Section 1.1 we illustrated the process of translating English sentences into logical expressions involving propositions and logical connectives. Now that we have discussed quantifiers, we can express a wider variety of English sentences using logical expressions. Doing so eliminates ambiguity and makes it possible to reason with these sentences. (Section 3.1 covers rules of inference for reasoning with logical expressions.)

The following examples show how to use logical operators and quantifiers to express English sentences, similar to the kind that occur frequently in mathematical statements, in logic programming, and in artificial intelligence.

**EXAMPLE 15**

Express the statements “Some student in this class has visited Mexico” and “Every student in this class has visited either Canada or Mexico” using quantifiers.

**Solution:** Let the universe of discourse for the variable \(x\) be the set of students in your class. Let \(M(x)\) be the statement “\(x\) has visited Mexico” and \(C(x)\) the statement “\(x\) has visited Canada.” The statement “Some student in this class has visited Mexico” can be written as \(\exists x M(x)\). The statement “Every student in this class has visited either Canada or Mexico” can be written as \(\forall x (C(x) \lor M(x))\) (assuming that the inclusive, rather than the exclusive, or is what is meant here).

**EXAMPLE 16**

Express the statement “Everyone has exactly one best friend” as a logical expression.

**Solution:** Let \(B(x, y)\) be the statement “\(y\) is the best friend of \(x\).” To translate the sentence in the example, note that it says that for every person \(x\) there is another person \(y\) such that \(y\) is the best friend of \(x\) and that if \(z\) is a person other than \(y\), then \(z\) is not the best friend of \(x\). Consequently, we can translate the sentence as

\[
\forall x \exists y \forall z ((B(x, y) \land (z \neq y)) \rightarrow \neg B(x, z)).
\]

**EXAMPLE 17**

Express the statement “If somebody is female and is a parent, then this person is someone’s mother” as a logical expression.

**Solution:** Let \(F(x)\) be the statement “\(x\) is female,” let \(P(x)\) be the statement “\(x\) is a parent,” and let \(M(x, y)\) be the statement “\(x\) is the mother of \(y\).” Since the statement in the example pertains to all people, we can write it symbolically as

\[
\forall x ((F(x) \land P(x)) \rightarrow \exists y M(x, y)).
\]

**EXAMPLE 18**

Use quantifiers to express the statement “There is a woman who has taken a flight on every airline in the world.”

**Solution:** Let \(P(w, f)\) be “\(w\) has taken \(f\)” and \(Q(f, a)\) be “\(f\) is a flight on \(a\).” We can express the statement as

\[
\exists w \forall a \exists f (P(w, f) \land Q(f, a)),
\]

where the universes of discourse for \(w, f, a\) consist of all the women in the world, all airplane flights, and all airlines, respectively.
The statement could also be expressed as
\[ \exists w \forall a \exists f \ R(w, f, a), \]
where \( R(w, f, a) \) is "\( w \) has taken \( f \) on \( a \)." Although this is more compact, it somewhat obscures the relationships between the variables. Consequently, the first solution is usually preferable.

As mentioned earlier, quantifiers are often used in the definition of mathematical concepts. One example that you may be familiar with is the concept of limit, which is important in calculus.

**EXAMPLE 19**

*(Calculus required)* Express the definition of a limit using quantifiers.

**Solution**: Recall that the definition of the statement
\[ \lim_{x \to a} f(x) = L \]
is: For every real number \( \epsilon > 0 \) there exists a real number \( \delta > 0 \) such that \( |f(x) - L| < \epsilon \) whenever \( 0 < |x - a| < \delta \). This definition of a limit can be phrased in terms of quantifiers by
\[ \forall \epsilon \exists \delta \forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon), \]
where the universe of discourse for the variables \( \delta \) and \( \epsilon \) is the set of positive real numbers and for \( x \) is the set of real numbers.

This definition can also be expressed as
\[ \forall \epsilon > 0 \exists \delta > 0 \forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \]
when the universe of discourse for the variables \( \epsilon \) and \( \delta \) is the set of all real numbers, rather than the set of positive real numbers.

**EXAMPLES FROM LEWIS CARROLL** *(optional)*

Lewis Carroll (really C. L. Dodgson writing under a pseudonym), the author of *Alice in Wonderland*, is also the author of several works on symbolic logic. His books contain many examples of reasoning using quantifiers. The next two examples come from his book *Symbolic Logic*; other examples from that book are given in the exercise set at the end of this section. These examples illustrate how quantifiers are used to express various types of statements.

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*Charles Lutwidge Dodgson (1832–1898).* We know Charles Dodgson as Lewis Carroll—the pseudonym he used in his writings on logic. Dodgson, the son of a clergyman, was the third of 11 children, all of whom stuttered. He was uncomfortable in the company of adults and is said to have spoken without stuttering only to young girls, many of whom he entertained, corresponded with, and photographed (often in the nude). Although attracted to young girls, he was extremely puritanical and religious. His friendship with the three young daughters of Dean Liddell led to his writing *Alice in Wonderland*, which brought him money and fame.

Dodgson graduated from Oxford in 1854 and obtained his master of arts degree in 1857. He was appointed lecturer in mathematics at Christ Church College, Oxford, in 1855. He was ordained in the Church of England in 1861 but never practiced his ministry. His writings include articles and books on geometry, determinants, and the mathematics of tournaments and elections. (He also used the pseudonym Lewis Carroll for his many works on recreational logic.)
Consider the following statements. The first two are called premises and the third is called the conclusion. The entire set is called an argument.

“All lions are fierce.”
“Some lions do not drink coffee.”
“Some fierce creatures do not drink coffee.”

(In Section 3.1 we will discuss the issue of determining whether the conclusion is a valid consequence of the premises. In this example, it is.) Let \( P(x), Q(x), \) and \( R(x) \) be the statements “\( x \) is a lion,” “\( x \) is fierce,” and “\( x \) drinks coffee,” respectively. Assuming that the universe of discourse is the set of all creatures, express the statements in the argument using quantifiers and \( P(x), Q(x), \) and \( R(x) \).

**Solution:** We can express these statements as:

\[
\forall x (P(x) \rightarrow Q(x)).
\]
\[
\exists x (P(x) \land \neg R(x)).
\]
\[
\exists x (Q(x) \land \neg R(x)).
\]

Notice that the second statement cannot be written as \( \exists x (P(x) \rightarrow \neg R(x)) \). The reason is that \( P(x) \rightarrow \neg R(x) \) is true whenever \( x \) is not a lion, so that \( \exists x (P(x) \rightarrow \neg R(x)) \) is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as

\[
\exists x (Q(x) \rightarrow \neg R(x)).
\]

Consider the following statements, of which the first three are premises and the fourth is a valid conclusion.

“All hummingbirds are richly colored.”
“No large birds live on honey.”
“Birds that do not live on honey are dull in color.”
“Hummingbirds are small.”

Let \( P(x), Q(x), R(x), \) and \( S(x) \) be the statements “\( x \) is a hummingbird,” “\( x \) is large,” “\( x \) lives on honey,” and “\( x \) is richly colored,” respectively. Assuming that the universe of discourse is the set of all birds, express the statements in the argument using quantifiers and \( P(x), Q(x), R(x), \) and \( S(x) \).

**Solution:** We can express the statements in the argument as:

\[
\forall x (P(x) \rightarrow S(x)).
\]
\[
\neg \exists x (Q(x) \land R(x)).
\]
\[
\forall x (\neg R(x) \rightarrow \neg S(x)).
\]
\[
\forall x (P(x) \rightarrow \neg Q(x)).
\]

(Note we have assumed that “small” is the same as “not large” and that “dull in color” is the same as “not richly colored.” To show that the fourth statement is a valid conclusion of the first three, we need to use rules of inference that will be discussed in Section 3.1.)

**BINDING VARIABLES**

When a quantifier is used on the variable \( x \) or when we assign a value to this variable, we say that this occurrence of the variable is bound. An occurrence of a variable that is not
bound by a quantifier or set equal to a particular value is said to be **free**. All the variables that occur in a propositional function must be bound to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

Many mathematical statements involve multiple quantifications of propositional functions involving more than one variable. It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers. These remarks are illustrated by Examples 22, 23, and 24. In each of these examples the universe of discourse for each variable is the set of real numbers.

**EXAMPLE 22**

Let $P(x, y)$ be the statement "$x + y = y + x$." What is the truth value of the quantification $\forall x \forall y \ P(x, y)$?

**Solution:** The quantification

$$\forall x \forall y \ P(x, y)$$

denotes the proposition

"For all real numbers $x$ and for all real numbers $y$, it is true that $x + y = y + x$."

Since $P(x, y)$ is true for all real numbers $x$ and $y$, the proposition $\forall x \forall y \ P(x, y)$ is true.

**EXAMPLE 23**

Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x \ Q(x, y)$ and $\forall x \exists y \ Q(x, y)$?

**Solution:** The quantification

$$\exists y \forall x \ Q(x, y)$$

denotes the proposition

"There is a real number $y$ such that for every real number $x$, $Q(x, y)$ is true."

No matter what value of $y$ is chosen, there is only one value of $x$ for which $x + y = 0$. Since there is no real number $y$ such that $x + y = 0$ for all real numbers $x$, the statement $\exists y \forall x \ Q(x, y)$ is false.

The quantification

$$\forall x \exists y \ Q(x, y)$$

denotes the proposition

"For every real number $x$ there is a real number $y$ such that $Q(x, y)$ is true."

Given a real number $x$, there is a real number $y$ such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y \ Q(x, y)$ is true.

Example 23 illustrates that the order in which quantifiers appear makes a difference. The statement $\exists y \forall x \ P(x, y)$ and $\forall x \exists y \ P(x, y)$ are not logically equivalent. The statement $\exists y \forall x \ P(x, y)$ is true if and only if there is a $y$ that makes $P(x, y)$ true for every $x$. So, for this statement to be true, there must be a particular value of $y$ for
which $P(x, y)$ is true regardless of the choice of $x$. On the other hand, $\forall x \exists y P(x, y)$ is true if and only if for every value of $x$ there is a value of $y$ for which $P(x, y)$ is true. So, for this statement to be true, no matter which $x$ you choose, there must be a value of $y$ (possibly depending on the $x$ you choose) for which $P(x, y)$ is true. In other words, in the second case $y$ can depend on $x$, whereas in the first case $y$ is a constant independent of $x$.

From these observations, it follows that if $\exists y \forall x P(x, y)$ is true, then $\forall x \exists y P(x, y)$ must also be true. However, if $\forall x \exists y P(x, y)$ is true, it is not necessary for $\exists y \forall x P(x, y)$ to be true. (See Supplementary Exercises 8 and 10 at the end of this chapter.)

In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops. (Of course, if there are infinitely many elements in the universe of discourse of some variable, we cannot actually loop through all values. Nevertheless, this way of thinking is helpful in understanding nested quantifiers.) For example, to see whether $\forall x \forall y P(x, y)$ is true, we loop through the values for $x$, and for each $x$ we loop through the values for $y$. If we find that $P(x, y)$ is true for all values for $x$ and $y$, we have determined that $\forall x \forall y P(x, y)$ is true. If we ever hit a value $x$ for which we hit a value $y$ for which $P(x, y)$ is false, we have shown that $\forall x \forall y P(x, y)$ is false.

Similarly, to determine whether $\forall x \exists y P(x, y)$ is true, we loop through the values for $x$. For each $x$ we loop through the values for $y$, and then we find a value $y$ for which $P(x, y)$ is true. If for all $x$ we hit such a $y$, then $\forall x \exists y P(x, y)$ is true; if for some $x$ we never hit such a $y$, then $\forall x \exists y P(x, y)$ is false.

To see whether $\exists y \forall x P(x, y)$ is true, we loop through the values for $x$ until we find an $x$ for which $P(x, y)$ is always true when we loop through all values for $y$. Once we find such an $x$, we know that $\exists y \forall x P(x, y)$ is true. If we never hit such an $x$, then we know that $\exists y \forall x P(x, y)$ is false.

Finally, to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for $x$, and for each $x$ we loop through the values for $y$ until we hit an $x$ for which we hit a $y$ for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an $x$ for which we hit a $y$ such that $P(x, y)$ is true.

Table 2 summarizes the meanings of the different possible quantifications involving two variables.

Quantifications of more than two variables are also common, as Example 24 illustrates.

<table>
<thead>
<tr>
<th>TABLE 2</th>
<th>Quantifications of Two Variables.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Statement</strong></td>
<td><strong>When True?</strong></td>
</tr>
<tr>
<td>$\forall x \forall y P(x, y)$</td>
<td>$P(x, y)$ is true for every pair $x, y$.</td>
</tr>
<tr>
<td>$\forall y \forall x P(x, y)$</td>
<td>For every $x$ there is a $y$ for which $P(x, y)$ is true.</td>
</tr>
<tr>
<td>$\forall x \exists y P(x, y)$</td>
<td>There is an $x$ for which $P(x, y)$ is true for every $y$.</td>
</tr>
<tr>
<td>$\exists x \forall y P(x, y)$</td>
<td>There is a pair $x, y$ for which $P(x, y)$ is true.</td>
</tr>
</tbody>
</table>
EXAMPLE 24

Let \( Q(x, y, z) \) be the statement "\( x + y = z \)." What are the truth values of the statements \( \forall x \forall y \exists z Q(x, y, z) \) and \( \exists z \forall x \forall y Q(x, y, z) \)?

Solution: Suppose that \( x \) and \( y \) are assigned values. Then, there exists a real number \( z \) such that \( x + y = z \). Consequently, the quantification

\[ \forall x \forall y \exists z Q(x, y, z), \]

which is the statement

"For all real numbers \( x \) and for all real numbers \( y \) there is a real number \( z \) such that \( x + y = z \),"

is true. The order of the quantification here is important, since the quantification

\[ \exists z \forall x \forall y Q(x, y, z), \]

which is the statement

"There is a real number \( z \) such that for all real numbers \( x \) and for all real numbers \( y \) it is true that \( x + y = z \),"

is false, since there is no value of \( z \) that satisfies the equation \( x + y = z \) for all values of \( x \) and \( y \).

NEGATIONS

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statement

"Every student in the class has taken a course in calculus."

This statement is a universal quantification, namely,

\[ \forall x P(x), \]

where \( P(x) \) is the statement "\( x \) has taken a course in calculus." The negation of this statement is "It is not the case that every student in the class has taken a course in calculus." This is equivalent to "There is a student in the class who has not taken a course in calculus." And this is simply the existential quantification of the negation of the original propositional function, namely,

\[ \exists x \neg P(x). \]

This example illustrates the following equivalence:

\[ \neg \forall x P(x) \iff \exists x \neg P(x). \]

Suppose we wish to negate an existential quantification. For instance, consider the proposition "There is a student in this class who has taken a course in calculus." This is the existential quantification

\[ \exists x Q(x), \]

where \( Q(x) \) is the statement "\( x \) has taken a course in calculus." The negation of this statement is the proposition "It is not the case that there is a student in this class who has taken a course in calculus." This is equivalent to "Every student in this class has not taken calculus," which is just the universal quantification of the negation of the
TABLE 3  Negating Quantifiers.

<table>
<thead>
<tr>
<th>Negation</th>
<th>Equivalent Statement</th>
<th>When Is Negation True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg \exists x P(x)$</td>
<td>$\forall x \neg P(x)$</td>
<td>$P(x)$ is false for every $x$.</td>
<td>There is an $x$ for which $P(x)$ is true.</td>
</tr>
<tr>
<td>$\neg \forall x P(x)$</td>
<td>$\exists x \neg P(x)$</td>
<td>There is an $x$ for which $P(x)$ is false.</td>
<td>$P(x)$ is true for every $x$.</td>
</tr>
</tbody>
</table>

original propositional function, or, phrased in the language of quantifiers,

$\forall x \neg Q(x)$.

This example illustrates the equivalence

$\neg \exists x Q(x) \iff \forall x \neg Q(x)$.

Negations of quantifiers are summarized in Table 3.

---

**Exercises**

1. Let $P(x)$ denote the statement “$x \leq 4$.” What are the truth values of the following?
   a) $P(0)$  
   b) $P(4)$  
   c) $P(6)$

2. Let $P(x)$ be the statement “the word $x$ contains the letter $a$.” What are the truth values of the following?
   a) $P(orange)$  
   b) $P(lemon)$  
   c) $P(true)$  
   d) $P(false)$

3. Let $Q(x, y)$ denote the statement “$x$ is the capital of $y$.” What are the truth values of the following?
   a) $Q(Denver, Colorado)$  
   b) $Q(Detroit, Michigan)$  
   c) $Q(Massachusetts, Boston)$  
   d) $Q(New York, New York)$

4. State the value of $x$ after the statement if $P(x)$ then $x = 1$ is executed, where $P(x)$ is the statement “$x > 1$,” if the value of $x$ when this statement is reached is
   a) $x = 0$  
   b) $x = 1$  
   c) $x = 2$

5. Let $P(x)$ be the statement “$x$ spends more than five hours every weekday in class.” where the universe of discourse for $x$ is the set of students. Express each of the following quantifications in English.
   a) $\exists x P(x)$  
   b) $\forall x P(x)$  
   c) $\exists x \neg P(x)$  
   d) $\forall x \neg P(x)$

6. Let $P(x, y)$ be the statement “$x$ has taken class $y$.” where the universe of discourse for $x$ is the set of all students in your class and for $y$ is the set of all computer science courses at your school. Express each of the following quantifications in English.
   a) $\exists x \exists y P(x, y)$  
   b) $\exists x \forall y P(x, y)$  
   c) $\forall x \exists y P(x, y)$  
   d) $\forall x \forall y P(x, y)$

7. Let $W(x, y)$ mean that $x$ has visited $y$, where the universe of discourse for $x$ is the set of all students in your school and the universe of discourse for $y$ is the set of all Web sites. Express each of the following statements by a simple English sentence.
   a) $W(Sarah Smith, www.att.com)$
   b) $\exists x W(x, www.imdb.org)$
   c) $\exists y W(Jose Orez, y)$
   d) $\exists y (W(Ashok Puri, y) \land W(Cindy Yoon, y))$
   e) $\exists y \forall z (y \neq (David Belcher) \land (W(David Belcher, z) \rightarrow W(y, z)))$
   f) $\exists y \forall z ((y \neq (x \neq y) \land (W(x, z) \leftrightarrow W(y, z)))$

8. Let $C(x, y)$ mean that $x$ is enrolled in $y$, where the universe of discourse for $x$ is the set of all students in your school and the universe of discourse for $y$ is the set of all classes being given at your school. Express each of the following statements by a simple English sentence.
   a) $C(Randy Goldberg, CS 252)$
   b) $\exists x C(x, Math 695)$
   c) $\exists y C(Carol Sitea, y)$
   d) $\exists x (C(x, Math 222) \land C(x, CS 252)))$
   e) $\exists x \forall y (C(x, y) \land (x \neq y) \rightarrow (C(x, z) \rightarrow C(y, z)))$
   f) $\exists x \forall y (C(x, y) \land (C(x, z) \leftrightarrow C(y, z)))$

9. Let $P(x)$ be the statement “$x$ can speak Russian” and let $Q(x)$ be the statement “$x$ knows the computer language C++.” Express each of the following sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. For the universe of discourse for quantifiers use the set of all students at your school.
   a) There is a student at your school who can speak Russian and who knows C++.
b) There is a student at your school who can speak Russian but who doesn’t know C++.
c) Every student at your school either can speak Russian or knows C++.
d) No student at your school can speak Russian or knows C++.

10. Let \( Q(x, y) \) be the statement “\( x \) has been a contestant on \( y \).” Express each of the following sentences in terms of \( Q(x, y) \), quantifiers, and logical connectives, where the universe of discourse for \( x \) is the set of all students at your school and for \( y \) is the set of all quiz shows on television.

a) There is a student at your school who has been a contestant on a television quiz show.
b) No student at your school has ever been a contestant on a television quiz show.
c) There is a student at your school who has been a contestant on \( \text{Jeopardy} \) and on \( \text{Wheel of Fortune} \).
d) Every television quiz show has had a student from your school as a contestant.
e) At least two students from your school have been contestants on \( \text{Jeopardy} \).

11. Let \( L(x, y) \) be the statement “\( x \) loves \( y \),” where the universe of discourse for both \( x \) and \( y \) is the set of all people in the world. Use quantifiers to express each of the following statements:

a) Everybody loves Jerry.
b) Everybody loves somebody.
c) There is somebody whom everybody loves.
d) Nobody loves everybody.
e) There is somebody whom Lydia does not love.
f) There is somebody whom no one loves.
g) There is exactly one person whom everybody loves.
h) There are exactly two people whom Lynn loves.
i) Everyone loves himself or herself.
j) There is someone who loves no one besides himself or herself.

12. Let \( F(x, y) \) be the statement “\( x \) can fool \( y \),” where the universe of discourse is the set of all people in the world. Use quantifiers to express each of the following statements:

a) Everybody can fool Fred.
b) Evelyn can fool everybody.
c) Everybody can fool somebody.
d) There is no one who can fool everybody.
e) Everyone can be fooled by somebody.
f) No one can fool both Fred and Jerry.
g) Nancy can fool exactly two people.
h) There is exactly one person whom everybody can fool.
i) No one can fool himself or herself.
j) There is someone who can fool exactly one person besides himself or herself.

13. Let \( S(x) \) be the predicate “\( x \) is a student,” \( F(x) \) the predicate “\( x \) is a faculty member,” and \( A(x, y) \) the predicate “\( x \) has asked \( y \) a question,” where the universe of discourse is the set of all people associated with your school. Use quantifiers to express each of the following statements.

a) Lois has asked Professor Michaels a question.
b) Every student has asked Professor Gross a question.
c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

14. Let \( I(x) \) be the statement “\( x \) has an Internet connection” and \( C(x, y) \) be the statement “\( x \) and \( y \) have chatted over the Internet,” where the universe of discourse for the variables \( x \) and \( y \) is the set of all students in your class. Use quantifiers to express each of the following statements.

a) Jerry does not have an Internet connection.
b) Rachel has not chatted over the Internet with Chelsea.
c) Jan and Sharon have never chatted over the Internet.
d) No one in the class has chatted with Bob.
e) Sanjay has chatted with everyone except Joseph.
f) Someone in your class does not have an Internet connection.
g) Not everyone in your class has an Internet connection.
h) Exactly one student in your class has an Internet connection.
i) Everyone except one student in your class has an Internet connection.

15. Let \( M(x, y) \) be “\( x \) has sent \( y \) an e-mail message” and \( T(x, y) \) be “\( x \) has telephoned \( y \),” where the universe of discourse is the set of all students in your class. Use quantifiers to express each of the following statements.
1.3 Exercises

(Alternate that all e-mail messages that were sent are received, which is not the way things often work.)

a) Chou has never sent an e-mail message to Koko.

b) Arlene has never sent an e-mail message to or telephoned Sarah.

c) Jose has never received an e-mail message from Deborah.

d) Every student in your class has sent e-mail message to Ken.

e) No one in your class has telephoned Nina.

f) Everyone in class has either telephoned Avi or sent him an e-mail message.

g) There is a student in your class who has sent everyone else in your class an e-mail message.

h) There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.

i) There are two students in your class who have sent each other e-mail messages.

j) There is a student who has sent himself or herself an e-mail message.

k) There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.

l) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.

m) There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student.

n) There are two students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.

16. Use quantifiers to express each of the following statements.

a) There is a student in this class who can speak Hindi.

b) Every student in this class knows how to drive a car.

c) Some student in this class has visited Alaska but has not visited Hawaii.

d) All students in this class have learned at least one programming language.

17. Use quantifiers to express the following statements.

a) Every computer science student needs a course in discrete mathematics.

b) There is a student in this class who owns a personal computer.

c) Every student in this class has taken at least one computer science course.

d) There is a student in this class who has taken at least one course in computer science.

e) Every student in this class has been in every building on campus.

f) There is a student in this class who has been in every room of at least one building on campus.

g) Every student in this class has been in at least one room of every building on campus.

18. A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of the following statements in terms of quantifiers and then determine its truth value.

a) There is a student in the class who is a junior.

b) Every student in the class is a computer science major.

c) There is a student in the class who is neither a mathematics major nor a junior.

d) Every student in the class is either a sophomore or a computer science major.

e) There is a major such that there is a student in the class in every year of study with that major.

19. Let \( P(x) \) be the statement "\( x = x^2 \)." If the universe of discourse is the set of integers, what are the truth values of the following?

a) \( P(0) \)

b) \( P(1) \)

c) \( P(2) \)

d) \( P(-1) \)

e) \( \exists x P(x) \)

f) \( \forall x P(x) \)

20. Let \( Q(x, y) \) be the statement "\( x + y = x - y \)." If the universe of discourse for both variables is the set of integers, what are the truth values of the following?

a) \( Q(1, 1) \)

b) \( Q(2, 0) \)

c) \( \forall y Q(1, y) \)

d) \( \exists y Q(x, y) \)

e) \( \exists x \exists y Q(x, y) \)

f) \( \forall x \exists y Q(x, y) \)

g) \( \exists y \forall x Q(x, y) \)

h) \( \forall y \exists x Q(x, y) \)

i) \( \forall x \forall y Q(x, y) \)

21. Determine the truth value of each of the following statements if the universe of discourse for all variables is the set of all integers.

\( a) \ \forall n (n^2 \geq 0) \)

\( b) \ \exists n \ (n^2 = 2) \)

\( c) \ \forall n (n \geq n) \)

\( d) \ \forall n \ \exists m \ (n < m) \)

\( e) \ \forall n \ \forall m \ (n < m^2) \)

\( f) \ \forall n \ \exists m \ (n + m = 0) \)

\( g) \ \forall n \ \exists m \ (n = m) \)

\( h) \ \exists n \ \exists m \ (m^2 + n = 5) \)

\( i) \ \forall n \ \exists m \ (n^2 + m = 6) \)

\( j) \ \exists n \ \exists m \ (n + m = 4 \land n - m = 1) \)

\( k) \ \exists n \ \exists m \ (n + m = 4 \land n - m = 2) \)

\( l) \ \forall n \ \exists m \ (p = (m + n)^2) \)

22. Determine the truth value of each of the following statements if the universe of discourse for each variable is the set of real numbers.

\( a) \ \exists x (x^2 = 2) \)

\( b) \ \exists x (x^2 = -1) \)

\( c) \ \forall x \exists y (x = y^2) \)

\( d) \ \forall x \exists y (x = y^2) \)

\( e) \ \exists x \forall y (x + y = 0) \)

\( f) \ \exists x \exists y (x + y = x + y) \)

\( g) \ \forall x \exists y (x = y + 1) \)

\( h) \ \exists x \forall y (x + y = 0) \)
i) $\forall x \exists y(x + y = 1)$  
ii) $\exists x \exists y(x + 2y = 2 \land 2x + 4y = 5)$  
iii) $\forall x \exists y(x - y = 2 \land 2x - y = 1)$  
iv) $\forall x \forall y \exists z = (x + y)/2$

23. Suppose the universe of discourse of the propositional function $P(x, y)$ consists of pairs $x$ and $y$, where $x$ is 1, 2, or 3 and $y$ is 1, 2, or 3. Write out the following propositions using disjunctions and conjunctions.

a) $\exists x \exists y P(x, y)$  
b) $\forall x \forall y P(x, y)$  
c) $\forall x \forall y P(x, y)$  
d) $\exists x \exists y P(x, y)$  
e) $\exists x \forall y P(x, y)$  
f) $\forall x \exists y P(x, y)$

24. Rewrite each of the following statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg \exists y \exists x P(x, y)$  
b) $\neg \forall x \exists y P(x, y)$  
c) $\neg \exists y (Q(y) \land \forall x \neg R(x, y))$  
d) $\neg \exists y \exists x (R(x, y) \land \forall x S(x, y))$  
e) $\neg \exists y \forall x \exists z (T(x, y, z) \land \exists x \forall z U(x, y, z))$

25. Rewrite each of the following statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg \forall x \forall y P(x, y)$  
b) $\neg \forall y \exists x P(x, y)$  
c) $\neg \forall x \forall y P(x, y)$  
d) $\neg (\exists x \exists y \neg P(x, y) \lor \forall x \forall y Q(x, y))$  
e) $\neg \forall x (\exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z))$

26. Express each of the following statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that."")

a) All dogs have fleas.  
b) No one has lost more than one thousand dollars playing the lottery.  
c) There is a student in this class who has chatted with exactly one other student.  
d) No student in this class has sent e-mail to exactly two other students in this class.  
e) Some student has solved every exercise in this book.  
f) No student has solved at least one exercise in every section of this book.

27. Express each of the following statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that."")

a) There is no dog that can talk.  
b) There is no one in this class who knows French and Russian.  
c) Every student in this class has taken exactly two mathematics classes at this school.

d) Someone has visited every country in the world except Libya.  
e) No one has climbed every mountain in the Himalayas.  
f) Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.

28. Express the negations of the following propositions using quantifiers, and in English.

a) Every student in this class likes mathematics.  
b) There is a student in this class who has never seen a computer.  
c) There is a student in this class who has taken every mathematics course offered at this school.  
d) There is a student in this class who has been in at least one room of every building on campus.

29. Use quantifiers to express the associative law for multiplication of real numbers.

30. Use quantifiers to express the distributive laws of multiplication over addition for real numbers.

Exercises 31–34 are based on questions found in the book Symbolic Logic by Lewis Carroll.

31. Let $P(x), Q(x),$ and $R(x)$ be the statements "$x$ is a professor," "$x$ is ignorant," and "$x$ is vain," respectively. Express each of the following statements using quantifiers, logical connectives, and $P(x), Q(x),$ and $R(x),$ where the universe of discourse is the set of all people.

a) No professors are ignorant.  
b) All ignorant people are vain.  
c) No professors are vain.  
d) Does (c) follow from (a) and (b)? If not, is there a correct conclusion?

32. Let $P(x), Q(x),$ and $R(x)$ be the statements "$x$ is a clear explanation," "$x$ is satisfactory," and "$x$ is an excuse," respectively. Suppose that the universe of discourse for $x$ is the set of all English text. Express each of the following statements using quantifiers, logical connectives, and $P(x), Q(x),$ and $R(x).$

a) All clear explanations are satisfactory.  
b) Some excuses are unsatisfactory.  
c) Some excuses are not clear explanations.  
d) Does (c) follow from (a) and (b)? If not, is there a correct conclusion?

33. Let $P(x), Q(x),$ and $R(x)$ be the statements "$x$ is a baby," "$x$ is logical," "$x$ is able to manage a crocodile," and "$x$ is despised," respectively. Suppose that the universe of discourse is the set of all people. Express each of the following statements using quantifiers, logical connectives, and $P(x), Q(x),$ and $R(x).$

a) Babies are illogical.  
b) Nobody is despised who can manage a crocodile.  
c) Illogical persons are despised.  
d) Babies cannot manage crocodiles.  
e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?
34. Let \( P(x) \), \( Q(x) \), \( R(x) \), and \( S(x) \) be the statements “\( x \) is a duck,” “\( x \) is one of my poultry,” “\( x \) is an officer,” and “\( x \) is willing to walts,” respectively. Express each of the following statements using quantifiers; logical connectives, and \( P(x) \), \( Q(x) \), \( R(x) \), and \( S(x) \).

a) No ducks are willing to walts.

b) No officers ever decline to walts.

c) All my poultry are ducks.

d) My poultry are not officers.

*e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

35. Show that the statements \( \neg \exists x \forall y P(x, y) \) and \( \forall x \exists y \neg P(x, y) \) have the same truth value.

36. Show that \( \forall x (P(x) \land Q(x)) \) and \( \forall x (P(x) \land \forall x Q(x)) \) have the same truth value.

37. Show that \( \exists x (P(x) \lor Q(x)) \) and \( \exists x P(x) \lor \exists x Q(x) \) have the same truth value.

38. Establish the following logical equivalences, where \( A \) is a proposition not involving any quantifiers.

\[
\begin{align*}
(\forall x P(x)) \land A &\iff \forall x (P(x) \land A) \\
(\exists x P(x)) \land A &\iff \exists x (P(x) \land A)
\end{align*}
\]

39. Establish the following logical equivalences, where \( A \) is a proposition not involving any quantifiers.

\[
\begin{align*}
\forall x (P(x) \lor A) &\iff \forall x (P(x) \lor A) \\
\exists x (P(x) \lor A) &\iff \exists x (P(x) \lor A)
\end{align*}
\]

40. Show that \( \forall x P(x) \land \forall x Q(x) \) and \( \forall x (P(x) \lor Q(x)) \) are not logically equivalent.

41. Show that \( \exists x P(x) \land \exists x Q(x) \) and \( \exists x (P(x) \lor Q(x)) \) are not logically equivalent.

*42. Show that \( \forall x P(x) \lor \forall x Q(x) \) and \( \forall x \forall y (P(x) \lor Q(y)) \) are logically equivalent. (The new variable \( y \) is used to combine the quantifications correctly.)

43. a) Show that \( \forall x (P(x) \lor \exists x Q(x)) \) and \( \forall x \exists y (P(x) \lor Q(y)) \) are equivalent.

b) Show that \( \forall x P(x) \lor \exists x Q(x) \) and \( \forall x \exists y (P(x) \lor Q(y)) \) are equivalent.

44. The notation \( \exists ! x P(x) \) denotes the proposition

“There exists a unique \( x \) such that \( P(x) \) is true.”

45. What are the truth values of the following statements?

\[
\begin{align*}
\exists ! x P(x) &\iff \exists x P(x) \\
\forall x P(x) &\iff \forall x P(x) \\
\exists x \neg P(x) &\iff \neg \exists x P(x)
\end{align*}
\]

46. Write out the quantification \( \exists ! x P(x) \), where the universe of discourse consists of the integers 1, 2, and 3, in terms of negations, conjunctions, and disjunctions.

47. Express the quantification \( \exists ! x P(x) \) using universal quantifications, existential quantifications, and logical operators.

A statement is in prenex normal form (PNF) if and only if it is of the form

\[
Q_1, x_2, \ldots, x_n \exists \phi(x_1, x_2, \ldots, x_n)
\]

where each \( Q_i \), \( i = 1, 2, \ldots, k \), is either the existential quantifier or the universal quantifier, and \( \phi(x_1, \ldots, x_n) \) is a predicate involving no quantifiers. For example, \( \exists y y(y(x) \land Q(y)) \) is in prenex normal form, whereas \( \exists x P(x) \lor \forall x Q(x) \) is not (since the quantifiers do not all occur first).

Every statement formed from propositional variables, predicates, \( T \) and \( F \) using logical connectives and quantifiers is equivalent to a statement in prenex normal form. Exercise 49 asks for a proof of this fact.

*48. Put the following statements in prenex normal form. (Hint: Use logical equivalence from Tables 5 and 6 in Section 1.2, Table 2 in this section, and Exercises 36–39 and 42–43 in this section.)

\[
\begin{align*}
(\exists x P(x) \land \forall x Q(x)) \lor A &\iff A \land (\exists x P(x) \land \forall x Q(x)) \\
(\forall x P(x) \lor A) \land \exists x Q(x) &\iff (\exists x Q(x) \lor A) \land (\forall x P(x)) \\
(\forall x P(x) \land \exists x Q(x)) \land \forall x R(x) &\iff (\forall x R(x) \land (\forall x P(x) \lor \exists x Q(x))) \lor A
\end{align*}
\]

*49. Show how to transform an arbitrary statement to a statement in prenex normal form that is equivalent to the given statement.

A real number \( x \) is called an upper bound of a set \( S \) of real numbers if \( x \) is greater than or equal to every member of \( S \). The real number \( x \) is called the least upper bound of a set \( S \) of real numbers if \( x \) is an upper bound of \( S \) and \( x \) is less than or equal to every upper bound of \( S \); if the least upper bound of a set \( S \) exists, it is unique.

50. a) Using quantifiers, express the fact that \( x \) is an upper bound of \( S \).

b) Using quantifiers, express the fact that \( x \) is the least upper bound of \( S \).

51. (Calculus required) Using quantifiers, express the fact that \( \lim_{n \to \infty} a_n = L \) does not exist.

The statement \( \lim_{x \to a} f(x) = L \) means that for every positive real number \( \epsilon \) there is a positive integer \( N \) such that \( |f(x) - L| < \epsilon \) whenever \( n > N \).

52. (Calculus required) Using quantifiers, express the statement that \( \lim_{n \to \infty} a_n = L \).

53. (Calculus required) Using quantifiers, express the statement that \( \lim_{n \to \infty} a_n = L \) does not exist.

54. (Calculus required) Using quantifiers to express the following definition: A sequence \( \{ a_n \} \) is a Cauchy sequence if for every real number \( \epsilon > 0 \) there exists a positive integer \( N \) such that \( |a_m - a_n| < \epsilon \) for every pair of positive integers \( m \) and \( n \) with \( m > N \) and \( n > N \).

55. (Calculus required) Using quantifiers and logical connectives to express the following definition: A number \( L \) is the limit superior of a sequence \( \{ a_n \} \) if for every real number \( \epsilon > 0 \) and not for infinitely many \( n \) and \( a_n > L + \epsilon \) for only finitely many \( n \).