1.5
Set Operations

INTRODUCTION

Two sets can be combined in many different ways. For instance, starting with the set of mathematics majors and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.

**Definition 1.** Let \( A \) and \( B \) be sets. The union of the sets \( A \) and \( B \), denoted by \( A \cup B \), is the set that contains those elements that are either in \( A \) or in \( B \), or in both.

An element \( x \) belongs to the union of the sets \( A \) and \( B \) if and only if \( x \) belongs to \( A \) or \( x \) belongs to \( B \). This tells us that \[ A \cup B = \{ x \mid x \in A \lor x \in B \}. \]

The Venn diagram shown in Figure 1 represents the union of two sets \( A \) and \( B \). The shaded area within the circle representing \( A \) or the circle representing \( B \) is the area that represents the union of \( A \) and \( B \).

We will give some examples of the union of sets.

**Example 1**

The union of the sets \( \{1, 3, 5\} \) and \( \{1, 2, 3\} \) is the set \( \{1, 2, 3, 5\} \); that is, \( \{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\} \).

**Example 2**

The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both).

**Definition 2.** Let \( A \) and \( B \) be sets. The intersection of the sets \( A \) and \( B \), denoted by \( A \cap B \), is the set containing those elements in both \( A \) and \( B \).

![Venn Diagram Representing the Union of A and B.](image1)

![Venn Diagram Representing the Intersection of A and B.](image2)
An element \( x \) belongs to the intersection of the sets \( A \) and \( B \) if and only if \( x \) belongs to \( A \) and \( x \) belongs to \( B \). This tells us that
\[
A \cap B = \{ x \mid x \in A \land x \in B \}.
\]
The Venn diagram shown in Figure 2 represents the intersection of two sets \( A \) and \( B \). The shaded area that is within both the circles representing the sets \( A \) and \( B \) is the area that represents the intersection of \( A \) and \( B \).
We give some examples of the intersection of sets.

**Example 3**
The intersection of the sets \{1, 3, 5\} and \{1, 2, 3\} is the set \{1, 3\}; that is, \{1, 3, 5\} \( \cap \) \{1, 2, 3\} = \{1, 3\}.

**Example 4**
The intersection of the set of all computer science majors at your school and the set of all mathematics majors is the set of all students who are joint majors in mathematics and computer science.

**Definition 3.** Two sets are called disjoint if their intersection is the empty set.

**Example 5**
Let \( A = \{1, 3, 5, 7, 9\} \) and \( B = \{2, 4, 6, 8, 10\} \). Since \( A \cap B = \emptyset \), \( A \) and \( B \) are disjoint.

We often are interested in finding the cardinality of the union of sets. To find the number of elements in the union of two finite sets \( A \) and \( B \), note that \(| A | + | B | \) counts each element that is in \( A \) but not in \( B \) or in \( B \) but not in \( A \) exactly once, and each element that is in both \( A \) and \( B \) exactly twice. Thus, if the number of elements that are in both \( A \) and \( B \) is subtracted from \(| A | + | B | \), elements in \( A \cap B \) will be counted only once. Hence,
\[
| A \cup B | = | A | + | B | - | A \cap B |.
\]
The generalization of this result to unions of an arbitrary number of sets is called the principle of inclusion–exclusion. The principle of inclusion–exclusion is an important technique used in the art of enumeration. We will discuss this principle and other counting techniques in detail in Chapters 4 and 5.

There are other important ways to combine sets.

**Definition 4.** Let \( A \) and \( B \) be sets. The difference of \( A \) and \( B \), denoted by \( A \setminus B \), is the set containing those elements that are in \( A \) but not in \( B \). The difference of \( A \) and \( B \) is also called the complement of \( B \) with respect to \( A \).

An element \( x \) belongs to the difference of \( A \) and \( B \) if and only if \( x \in A \) and \( x \notin B \). This tells us that
\[
A \setminus B = \{ x \mid x \in A \land x \notin B \}.
\]
The Venn diagram shown in Figure 3 represents the difference of the sets \( A \) and \( B \). The shaded area inside the circle that represents \( A \) and outside the circle that represents \( B \) is the area that represents \( A \setminus B \).

We give some examples of differences of sets.
EXAMPLE 6

The difference of \{1, 3, 5\} and \{1, 2, 3\} is the set \{5\}; that is, \{1, 3, 5\} \setminus \{1, 2, 3\} = \{5\}. This is different from the difference of \{1, 2, 3\} and \{1, 3, 5\}, which is the set \{2\}.

EXAMPLE 7

The difference of the set of computer science majors at your school and the set of mathematics majors at your school is the set of all computer science majors at your school who are not also mathematics majors.

Once the universal set \(U\) has been specified, the complement of a set can be defined.

**DEFINITION 5.** Let \(U\) be the universal set. The complement of the set \(A\), denoted by \(\overline{A}\), is the complement of \(A\) with respect to \(U\). In other words, the complement of the set \(A\) is \(U \setminus A\).

An element belongs to \(\overline{A}\) if and only if \(x \notin A\). This tells us that

\[\overline{A} = \{ x \mid x \notin A \}\]

In Figure 4 the shaded area outside the circle that represents \(A\) is the area representing \(\overline{A}\).

We give some examples of the complement of a set.

EXAMPLE 8

Let \(A = \{a, e, i, o, u\}\) (where the universal set is the set of letters of the English alphabet). Then \(\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}\).

EXAMPLE 9

Let \(A\) be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then \(A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\).

**SET IDENTITIES**

Table 1 lists the most important set identities. We will prove several of these identities here, using three different methods. These methods are presented to illustrate that there are often many different approaches to the solution of a problem. The proofs of
### Table 1: Set Identities

<table>
<thead>
<tr>
<th>Identity</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cup \emptyset = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cap U = A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup U = U$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$A \cap \emptyset = \emptyset$</td>
<td></td>
</tr>
<tr>
<td>$A \cup A = A$</td>
<td>Idempotent laws</td>
</tr>
<tr>
<td>$A \cap A = A$</td>
<td></td>
</tr>
<tr>
<td>$\overline{A} = A$</td>
<td>Complementation law</td>
</tr>
<tr>
<td>$A \cup B = B \cup A$</td>
<td>Commutative laws</td>
</tr>
<tr>
<td>$A \cap B = B \cap A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cup C) = (A \cup B) \cup C$</td>
<td>Associative laws</td>
</tr>
<tr>
<td>$A \cap (B \cap C) = (A \cap B) \cap C$</td>
<td></td>
</tr>
<tr>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
<td></td>
</tr>
<tr>
<td>$A \cup B = \overline{A} \cap \overline{B}$</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>$A \cap B = \overline{A} \cup \overline{B}$</td>
<td></td>
</tr>
</tbody>
</table>

The remaining identities will be left as exercises. The reader should note the similarity between these set identities and the logical equivalences discussed in Section 1.2. In fact, the set identities given can be proved directly from the corresponding logical equivalences. Furthermore, both are special cases of identities that hold for Boolean algebra (discussed in Chapter 9).

One way to prove that two sets are equal is to show that one of the sets is a subset of the other and vice versa. We illustrate this type of proof by establishing the second of De Morgan’s laws.

**Example 10**

Prove that $A \cap B = \overline{A} \cup \overline{B}$ by showing that each set is a subset of the other.

**Solution:** First, suppose that $x \in A \cap B$. It follows that $x \notin A \cap B$. This implies that $x \notin A$ or $x \notin B$. Hence, $x \in \overline{A}$ or $x \in \overline{B}$. Thus, $x \in A \cup B$. This shows that $A \cap B \subseteq A \cup B$.

Now suppose that $x \in A \cup B$. Then $x \in A$ or $x \in B$. It follows that $x \notin A$ or $x \notin B$. Hence, $x \notin A \cap B$. Therefore, $x \notin A \cap B$. This demonstrates that $A \cup B \subseteq A \cap B$. Since we have demonstrated that each set is a subset of the other, these two sets must be equal and the identity is proved.

Another way to verify set identities is to use set builder notation and the rules of logic. Consider the following proof of the second of De Morgan’s laws.
EXAMPLE 11

Use set builder notation and logical equivalences to show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: The following chain of equalities provides a demonstration of this identity:

$$A \cap B = \{ x \mid x \notin A \cap B \}$$
$$= \{ x \mid \neg (x \in (A \cap B)) \}$$
$$= \{ x \mid \neg (x \in A \land x \in B) \}$$
$$= \{ x \mid x \notin A \lor x \notin B \}$$
$$= \{ x \mid x \in \overline{A} \lor x \in \overline{B} \}$$
$$= \{ x \mid x \in \overline{A} \cup \overline{B} \}.$$

Note that the second De Morgan's law for logical equivalences was used in the fourth equality of this chain.

Set identities can also be proved using membership tables. We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used. (The reader should note the similarity between membership tables and truth tables.)

EXAMPLE 12

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: The membership table for these combinations of sets is shown in Table 2. This table has eight rows. Since the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid.

Additional set identities can be established using those that we have already proved. Consider the following example.

EXAMPLE 13

Let $A$, $B$, and $C$ be sets. Show that $A \cup (B \cap C) = (C \cup B) \cap A$.

![Table 2: A Membership Table for the Distributive Property.](image)
The fourth identity:

Each component of a union or intersection of a finite number of sets is a set.

The reader is directed to Table 2.

\( A \cup (A \cap C) \)

The commutative law for unions and intersections is proved.

\begin{figure}
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{example14a}
\caption{\( A \cup B \cup C \) is shaded.}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{example14b}
\caption{\( A \cap B \cap C \) is shaded.}
\end{subfigure}
\caption{(a) \( A \cup B \cup C \) is Shaded. (b) \( A \cap B \cap C \) is Shaded. The Union and Intersection of \( A, \ B, \) and \( C \).}
\end{figure}

**Solution:** We have
\[
\begin{align*}
A \cup (B \cap C) &= \overline{A} \cap (\overline{B} \cap C) \quad \text{by the first De Morgan's law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{by the second De Morgan's law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} \quad \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} \quad \text{by the commutative law for unions.}
\end{align*}
\]

**GENERALIZED UNIONS AND INTERSECTIONS**

Since unions and intersections of sets satisfy associative laws, the sets \( A \cup B \cup C \) and \( A \cap B \cap C \) are well defined when \( A, B, \) and \( C \) are sets. Note that \( A \cup B \cup C \) contains those elements that are in at least one of the sets \( A, B, \) and \( C \), and that \( A \cap B \cap C \) contains those elements that are in all of \( A, B, \) and \( C \). These combinations of the three sets, \( A, B, \) and \( C, \) are shown in Figure 5.

**EXAMPLE 14**

Let \( A = \{0, 2, 4, 6, 8\} \), \( B = \{0, 1, 2, 3, 4\} \), and \( C = \{0, 3, 6, 9\} \). What are \( A \cup B \cup C \) and \( A \cap B \cap C \)?

**Solution:** The set \( A \cup B \cup C \) contains those elements in at least one of \( A, B, \) and \( C \). Hence,
\[
A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.
\]

The set \( A \cap B \cap C \) contains those elements in all three of \( A, B, \) and \( C \). Thus,
\[
A \cap B \cap C = \{0\}.
\]

We can also consider unions and intersections of an arbitrary number of sets. We use the following definitions.

**DEFINITION 6.** The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.
We use the notation
\[ A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i \]
to denote the union of the sets \( A_1, A_2, \ldots, A_n \).

**DEFINITION 7.** The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation
\[ A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i \]
to denote the intersection of the sets \( A_1, A_2, \ldots, A_n \). We illustrate generalized unions and intersections with the following example.

**EXAMPLE 15**

Let \( A_i = \{i, i + 1, i + 2, \ldots\} \). Then
\[ \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{1, 2, 3, \ldots\}, \]
and
\[ \bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{n, n + 1, n + 2, \ldots\}. \]

**COMPUTER REPRESENTATION OF SETS**

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, since each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set \( U \) is finite (and of reasonable size so that the number of elements of \( U \) is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of \( U \), for instance \( a_1, a_2, \ldots, a_n \). Represent a subset \( A \) of \( U \) with the bit string of length \( n \), where the \( i \)th bit in this string is 1 if \( a_i \) belongs to \( A \) and is 0 if \( a_i \) does not belong to \( A \). The following example illustrates this technique.

**EXAMPLE 16**

Let \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), and the ordering of elements of \( U \) has the elements in increasing order; i.e., \( a_i = i \). What bit strings represent the subset of all odd integers in \( U \), the subset of all even integers in \( U \), and the subset of integers not exceeding 5 in \( U \)?
EXAMPLE 17

What is the bit string for the complement of the set \( \{3, 5, 7, 9\} \)?

\[ \begin{align*}
\text{set} & \quad \text{bit string} \\
\{3, 5, 7, 9\} & \quad 10101010 \\
\{1, 2, 3, 4\} & \quad 11111111 \\
\{6, 7, 8\} & \quad 00000000 \\
\{5, 6, 7\} & \quad 11111101 \\
\{1, 2, 3, 4, 5, 7\} & \quad 11010101
\end{align*} \]

We have seen that the bit string for the set \( \{1, 2, 3, 4, 5\} \) is represented by the string

\[ 11111101, \]

which corresponds to the set \( \{5, 6, 7\} \).

The bit strings for the sets \( \{1, 2, 3, 4\} \) and \( \{1, 2, 3, 4, 5, 7\} \) are 11111111 and 11010101, respectively. Use bit strings to find the union and intersection of these sets.

To obtain the bit string for the union and intersection of two sets, we perform bitwise Boolean operations on the bit strings representing the two sets. The bit in the ith position of the bit string for the union is 1 if either of the bits in the ith position of the bit strings for the two sets is 1; otherwise, it is 0. The bit in the ith position of the bit string for the intersection is 1 if both of the bits in the corresponding position of the bit strings are both 1; otherwise, it is 0. Hence, the bit string for the union of the two sets is 11011111, and the bit string for the intersection is 11010101.

EXAMPLE 18

The set of all integers in \( U \) that do not exceed 5, namely \( \{1, 3, 5, 7, 9\} \), is represented by the string 10101010. What is the bit string for the complement of this set?

The set of all integers in \( U \) that do not exceed 5, namely \( \{1, 3, 5, 7, 9\} \), is represented by the string 10101010. The bit string for the complement of this set is obtained by replacing 1 with 0 and vice versa. This yields the string 01010010, which corresponds to the set \( \{2, 4, 6, 8, 10\} \).

Solution: The bit string that represents the set of odd integers in \( U \), namely \( \{1, 3, 5, 7, 9\} \), has one bit in the first, third, seventh, and ninth positions, and a zero elsewhere.
Solution: The bit string for the union of these sets is
\[ 111100000 \lor 1010101010 = 111101010, \]
which corresponds to the set \{1, 2, 3, 4, 5, 7, 9\}. The bit string for the intersection of these sets is
\[ 111100000 \land 1010101010 = 101010000, \]
which corresponds to the set \{1, 3, 5\}.

Exercises

1. Let \( A \) be the set of students who live within one mile of school and let \( B \) be the set of students who walk to classes. Describe the students in each of the following sets.
   a) \( A \cap B \)
   b) \( A \cup B \)
   c) \( A - B \)
   d) \( B - A \)

2. Suppose that \( A \) is the set of sophomores at your school and \( B \) is the set of students in discrete mathematics at your school. Express each of the following sets in terms of \( A \) and \( B \).
   a) the set of sophomores taking discrete mathematics in your school
   b) the set of sophomores at your school who are not taking discrete mathematics
   c) the set of students at your school who either are sophomores or are taking discrete mathematics
   d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{0, 3, 6\} \). Find
   a) \( A \cup B \)
   b) \( A \cap B \)
   c) \( A - B \)
   d) \( B - A \)

4. Let \( A = \{a, b, c, d, e\} \) and \( B = \{a, b, c, d, e, f, g, h\} \). Find
   a) \( A \cup B \)
   b) \( A \cap B \)
   c) \( A - B \)
   d) \( B - A \)

5. Let \( A \) be a set. Show that \( \overline{A} = A \).

6. Let \( A \) be a set. Show that
   a) \( A \cup \emptyset = A \)
   b) \( A \cap \emptyset = \emptyset \)
   c) \( A \cup A = A \)
   d) \( A \cap A = A \)
   e) \( A - A = A \)
   f) \( A \cup U = U \)
   g) \( A \cap U = A \)
   h) \( \emptyset - A = \emptyset \)

7. Let \( A \) and \( B \) be sets. Show that
   a) \( A \cup B = B \cup A \)
   b) \( A \cap B = B \cap A \)

8. Find the sets \( A \) and \( B \) if \( A - B = \{1, 5, 7, 8\} \) and \( B - A = \{2, 10\} \) and \( A \cap B = \{3, 6, 9\} \).

9. Show that if \( A \) and \( B \) are sets, then \( A \cup B = \overline{\overline{A} \cap \overline{B}} \).
   a) by showing each side is a subset of the other side
   b) using a membership table

10. Let \( A \) and \( B \) be sets. Show that
    a) \( A \cap B \subseteq A \)
    b) \( A \subseteq (A \cup B) \)
    c) \( A - B \subseteq A \)
    d) \( A \cap (B - A) = \emptyset \)
    e) \( A \cup (B - A) = A \cup B \)

11. Show that if \( A, B, \) and \( C \) are sets, then \( \overline{A \cap B \cap C} = A \cup B \cup C \).
   a) by showing each side is a subset of the other side
   b) using a membership table

12. Let \( A, B, \) and \( C \) be sets. Show that
    a) \( (A \cup B) \subseteq (A \cup B \cup C) \)
    b) \( (A \cap B \cap C) \subseteq (A \cap B) \)
    c) \( (A - B) - C \subseteq A - C \)
    d) \( (A - C) \cap (C - B) = \emptyset \)
    e) \( (A - A) \cup (C - A) = (B \cup C) - A \)

13. Show that if \( A \) and \( B \) are sets, then \( A - B = A \cap \overline{B} \).

14. Show that if \( A \) and \( B \) are sets, then \( (A \cap B) \cup (A \cap B) = A \).

15. Let \( A, B, \) and \( C \) be sets. Show that
    a) \( A \cup (B \cup C) = (A \cup B) \cup C \)
    b) \( A \cap (B \cap C) = (A \cap B) \cap C \)
    c) \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)

16. Let \( A, B, \) and \( C \) be sets. Show that \( (A - B) - C = (A - C) - (B - C) \).

17. Let \( A = \{0, 2, 4, 6, 8, 10\} \) and \( B = \{0, 1, 2, 3, 4, 5, 6\} \), and \( C = \{4, 5, 6, 7, 8, 9, 10\} \). Find
    a) \( A \cap B \cap C \)
    b) \( A \cup B \cup C \)
    c) \( (A \cap B) \cap C \)
    d) \( (A \cap B) \cup C \)

18. Draw the Venn diagrams for each of the following combinations of the sets \( A, B, \) and \( C \).
    a) \( A \cap (B \cup C) \)
    b) \( A \cap B \cap C \)
    c) \( A \cup (B - A) \cup (B - C) \)

19. What can you say about the sets \( A \) and \( B \) if the following are true?
    a) \( A \cup B = A \)
    b) \( A \cap B = A \)
    c) \( A - B = A \)
    d) \( A \cap B = B \cap A \)
    e) \( A - B = B - A \)

20. Can you conclude that \( A = B \) if \( A, B, \) and \( C \) are sets such that
    a) \( A \cup C = B \cup C \)
    b) \( A \cap C = B \cap C \)

21. Let \( A \) and \( B \) be subsets of a universal set \( U \). Show that \( A \subseteq B \) if and only if \( B \subseteq A \).
The symmetric difference of $A$ and $B$, denoted by $A \oplus B$, is the set containing those elements in either $A$ or $B$, but not in both $A$ and $B$.

22. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.
23. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.
24. Draw a Venn diagram for the symmetric difference of the sets $A$ and $B$.
25. Show that $A \oplus B = (A \cup B) - (A \cap B)$.
26. Show that $A \oplus B = (A - B) \cup (B - A)$.
27. Show that if $A$ is a subset of a universal set $U$, then
   a) $A \oplus A = \emptyset$.
   b) $A \oplus \emptyset = A$.
   c) $A \oplus U = A$.
   d) $A \oplus A = U$.
28. Show that if $A$ and $B$ are sets, then
   a) $A \oplus B = B \oplus A$.
   b) $(A \oplus B) \oplus B = A$.
29. What can you say about the sets $A$ and $B$ if $A \oplus B = A$?
30. Determine whether the symmetric difference is associative; that is, if $A$, $B$, and $C$ are sets, does it follow that
   $$A \oplus (B \oplus C) = (A \oplus B) \oplus C?$$
31. Suppose that $A$, $B$, and $C$ are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?
32. If $A$, $B$, $C$, and $D$ are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?
33. If $A$, $B$, $C$, and $D$ are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?
34. Show that if $A$, $B$, and $C$ are sets, then
   $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$$
   $$- |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$ (This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 5.)
35. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find
   a) $\bigcup_{i=1}^{n} A_i$.
   b) $\bigcap_{i=1}^{n} A_i$.
36. Let $A_i = \{i, i+1, i+2, \ldots\}$. Find
   a) $\bigcup_{i=1}^{n} A_i$.
   b) $\bigcap_{i=1}^{n} A_i$.
37. Let $A_i$ be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding $i$. Find
   a) $\bigcup_{i=1}^{n} A_i$.
   b) $\bigcap_{i=1}^{n} A_i$.
38. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of the following sets with bit strings where the $i$th bit in the string is 1 if $i$ is in the set and 0 otherwise.
   a) $\{3, 4, 5\}$
   b) $\{1, 3, 6, 10\}$
   c) $\{2, 3, 4, 7, 8, 9\}$
39. Using the same universal set as in the last problem, find the set specified by each of the following bit strings.
   a) 11 1100 1111
   b) 01 0111 1000
   c) 10 0000 0001
40. What subsets of a finite universal set do the following bit strings represent?
   a) the string with all zeros
   b) the string with all ones
41. What is the bit string corresponding to the difference of two sets?
42. What is the bit string corresponding to the symmetric difference of two sets?
43. Show how bitw=wise operations on bit strings can be used to find the following combinations of $A = \{a, b, c, d, e, f\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, i, n, o, t, u, x, y\}$.
   a) $A \cup B$
   b) $A \cap B$
   c) $(A \cup D) \cap (B \cup C)$
   d) $A \cup B \cup C \cup D$
44. How can the union and intersection of $n$ sets that all are subsets of the universal set $U$ be found using bit strings?
45. The successor of the set $A$ is the set $A \cup \{A\}$. Find the successors of the following sets.
   a) $\{1, 2, 3\}$
   b) $\emptyset$
   c) $\{2\}$
   d) $\emptyset, \{2\}$
46. How many elements does the successor of a set with $n$ elements have?

Sometimes the number of times that an element occurs in an unordered collection matters. Multisets are unordered collections of elements where an element can occur as a member more than once. The notation $\{m_1, a_1, m_2, a_2, \ldots, m_r, a_r\}$ denotes the multiset with element $a_1$ occurring $m_1$ times, element $a_2$ occurring $m_2$ times, and so on. The numbers $m_i$, $i = 1, 2, \ldots, r$ are called the multiplicities of the elements $a_i$, $i = 1, 2, \ldots, r$.

Let $P$ and $Q$ be multisets. The union of the multisets $P$ and $Q$ is the multiset where the multiplicity of an element is the maximum of its multiplicities in $P$ and $Q$. The intersection of $P$ and $Q$ is the multiset where the multiplicity of an element is the minimum of its multiplicities in $P$ and $Q$. The difference of $P$ and $Q$ is the multiset where the multiplicity of an element is the multiplicity of the element in $P$ less its multiplicity in $Q$ unless this difference is negative, in which case the multiplicity is 0. The sum of $P$ and $Q$ is the multiset where the multiplicity of an element is the sum of its multiplicities in $P$ and $Q$. The union, intersection, and difference of $P$ and $Q$ are denoted by $P \cup Q$, $P \cap Q$, and $P - Q$, respectively (where these operations should not be confused with the analogous operations for sets). The sum of $P$ and $Q$ is denoted by $P + Q$.

47. Let $A$ and $B$ be the multisets $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ and $\{2 \cdot a, 3 \cdot b, 4 \cdot d\}$, respectively. Find
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B - A$
   e) $A + B$
48. Suppose that $A$ is the multiset that has as its elements the types of computer equipment needed by one
department of a university where the multiplicities are the number of pieces of each type needed, and \( B \) is the analogous multiset for a second department of the university. For instance, \( A \) could be the multiset \( \{107 \cdot \text{personal computers, } 44 \cdot \text{routers, } 6 \cdot \text{servers}\} \) and \( B \) could be the multiset \( \{14 \cdot \text{personal computers, } 6 \cdot \text{routers, } 2 \cdot \text{mainframes}\} \).

a) What combination of \( A \) and \( B \) represents the equipment the university should buy assuming both departments use the same equipment?

b) What combination of \( A \) and \( B \) represents the equipment that will be used by both departments if both departments use the same equipment?

c) What combination of \( A \) and \( B \) represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?

d) What combination of \( A \) and \( B \) represents the equipment that the university should purchase if the departments do not share equipment?

**Fuzzy sets** are used in artificial intelligence. Each element in the universal set \( U \) has a degree of membership, which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set \( S \). The fuzzy set \( S \) is denoted by listing the elements with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write \( \{0.6 \text{ Alice, } 0.9 \text{ Brian, } 0.4 \text{ Fred, } 0.1 \text{ Oscar, } 0.5 \text{ Rita}\} \) for the set \( F \) (of famous people) to indicate that Alice has a 0.6 degree of membership in \( F \), Brian has a 0.9 degree of membership in \( F \), Fred has a 0.4 degree of membership in \( F \), Oscar has a 0.1 degree of membership in \( F \), and Rita has a 0.5 degree of membership in \( F \) (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that \( R \) is the set of rich people with \( R = \{0.4 \text{ Alice, } 0.8 \text{ Brian, } 0.2 \text{ Fred, } 0.9 \text{ Oscar, } 0.7 \text{ Rita}\} \).

49. The **complement** of a fuzzy set \( S \) is the set \( \overline{S} \), with the degree of the membership of an element in \( \overline{S} \) equal to 1 minus the degree of membership of this element in \( S \). Find \( \overline{F} \) (the fuzzy set of people who are not famous) and \( \overline{R} \) (the fuzzy set of people who are not rich).

50. The **union** of two fuzzy sets \( S \) and \( T \) is the fuzzy set \( S \cup T \), where the degree of membership of an element in \( S \cup T \) is the maximum of the degrees of membership of this element in \( S \) and in \( T \). Find the fuzzy set \( F \cup R \) of rich or famous people.

51. The **intersection** of two fuzzy sets \( S \) and \( T \) is the fuzzy set \( S \cap T \), where the degree of membership of an element in \( S \cap T \) is the minimum of the degrees of membership of this element in \( S \) and in \( T \). Find the fuzzy set \( F \cap R \) of rich and famous people.

### 1.6 Functions

**INTRODUCTION**

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first). For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set \( \{A, B, C, D, F\} \). And suppose that the grades are \( A \) for Adams, \( C \) for Chou, \( B \) for Goodfriend, \( A \) for Rodriguez, and \( F \) for Stevens. This assignment of grades is illustrated in Figure 1.

![Assignment of Grades in a Discrete Mathematics Class](image.png)