1.4
Sets

INTRODUCTION

We will study a wide variety of discrete structures in this book. These include relations, which consist of ordered pairs of elements; combinations, which are unordered collections of elements; and graphs, which are sets of vertices and edges connecting vertices. Moreover, we will illustrate how these and other discrete structures are used in modeling and problem solving. In particular, many examples of the use of discrete structures in the storage, communication, and manipulation of data will be described. In this section we study the fundamental discrete structure upon which all other discrete structures are built, namely, the set.

Sets are used to group objects together. Often, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such collections in an organized fashion.

Note that the term object has been used without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated by the German mathematician Georg Cantor in 1895. The theory that results from this intuitive definition of a set leads to paradoxes, or logical inconsistencies, as the English philosopher Bertrand Russell showed in 1902 (see Exercise 26 for a description of one of these paradoxes). These logical inconsistencies can be avoided by building set theory starting with basic assumptions, called axioms. We will use Cantor’s original version of set theory, known as naive set theory, without developing an axiomatic version of set theory, since all sets considered in this book can be treated consistently using Cantor’s original theory.

We now proceed with our discussion of sets.

DEFINITION 1. The objects in a set are also called the elements, or members, of the set. A set is said to contain its elements.

Georg Cantor (1845–1918). Georg Cantor was born in St. Petersburg, Russia, where his father was a successful merchant. Cantor developed his interest in mathematics in his teens. He began his university studies in Zurich in 1862, but when his father died he left Zurich. He continued his university studies at the University of Berlin in 1863, where he studied under the eminent mathematicians Weierstrass, Kummer, and Kronecker. He received his doctor’s degree in 1867 after having written a dissertation on number theory. Cantor assumed a position at the University of Halle in 1869, where he continued working until his death.

Cantor is considered the founder of set theory. His contributions in this area include the discovery that the set of real numbers is uncountable. He is also noted for his many important contributions to analysis. Cantor also was interested in philosophy and wrote papers relating his theory of sets with metaphysics.

Cantor married in 1874 and had five children. His melancholy temperament was balanced by his wife’s happy disposition. Although he received a large inheritance from his father, he was poorly paid as a professor. To mitigate this, he tried to obtain a better-paying position at the University of Berlin. His appointment there was blocked by Kronecker, who did not agree with Cantor’s views on set theory. Cantor suffered from mental illness throughout the later years of his life. He died in 1918 in a psychiatric clinic.
There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation \( \{a, b, c, d\} \) represents the set with the four elements \( a, b, c, \) and \( d \).

**Example 1**

The set \( V \) of all vowels in the English alphabet can written as \( V = \{a, e, i, o, u\} \).

**Example 2**

The set \( O \) of odd positive integers less than 10 can be expressed by \( O = \{1, 3, 5, 7, 9\} \).

**Example 3**

Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, \( \{a, 2, \text{Fred}, \text{New Jersey}\} \) is the set containing the four elements \( a, 2, \) Fred, and New Jersey.

**Example 4**

Uppercase letters are usually used to denote sets. The boldface letters \( \mathbb{N}, \mathbb{Z}, \) and \( \mathbb{R} \) will be reserved to represent the set of natural numbers (0, 1, 2, 3, \ldots), the set of integers \( \ldots, -2, -1, 0, 1, 2, \ldots \), and the set of real numbers, respectively. We will occasionally use the notation \( \mathbb{Z}^+ \) to denote the set of positive integers. (Some people do not consider 0 a natural number, so be careful to check how the term *natural numbers* is used when you read other books.)

Sometimes the brace notation is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (\ldots) are used when the general pattern of the elements is obvious.

The set of positive integers less than 100 can be denoted by \( \{1, 2, 3, \ldots, 99\} \).

Since many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

**Bertrand Russell (1872–1970)**

Bertrand Russell was born into a prominent English family active in the progressive movement and having a strong commitment to liberty. He became an orphan at an early age and was placed in the care of his father's parents, who had him educated at home. He entered Trinity College, Cambridge, in 1890, where he excelled in mathematics and in moral science. He won a fellowship on the basis of his work on the foundations of geometry. In 1910 Trinity College appointed him to a lectureship in logic and the philosophy of mathematics.

Russell fought for progressive causes throughout his life. He held strong pacifist views, and his protests against World War I led to dismissal from his position at Trinity College. He was imprisoned for 6 months in 1918 because of an article he wrote that was branded as seditious. Russell fought for women's suffrage in Great Britain. In 1961, at the age of 89, he was imprisoned for the second time for his protests advocating nuclear disarmament.

Russell's greatest work was in his development of principles that could be used as a foundation for all of mathematics. His most famous work is *Principia Mathematica*, written with Alfred North Whitehead, which attempts to deduce all of mathematics using a set of primitive axioms. He wrote many books on philosophy, physics, and his political ideas. Russell won the Nobel Prize for literature in 1950.
DEFINITION 2. Two sets are equal if and only if they have the same elements.

Example 5.

The sets \{1, 3, 5\} and \{3, 5, 1\} are equal, since they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so that \{1, 3, 3, 5, 5, 5\} is the same as the set \{1, 3, 5\} since they have the same elements.

Another way to describe a set is to use set builder notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set \(O\) of all odd positive integers less than 10 can be written as

\[ O = \{x \mid x \text{ is an odd positive integer less than 10}\}. \]

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set of all real numbers can be written as

\[ R = \{x \mid x \text{ is a real number}\}. \]

Sets can also be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams the universal set \(U\), which contains all the objects under consideration, is represented by a rectangle. Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set. Venn diagrams are often used to indicate the relationships between sets. We show how a Venn diagram can be used in the following example.

Example 6.

Draw a Venn diagram that represents \(V\), the set of vowels in the English alphabet.

Solution: We draw a rectangle to indicate the universal set \(U\), which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent \(V\). Inside this circle we indicate the elements of \(V\) with points (see Figure 1).

We will now introduce notation used to describe membership in sets. We write \(a \in A\) to denote that \(a\) is an element of the set \(A\). The notation \(a \notin A\) denotes that \(a\) is not a member of the set \(A\). Note that lowercase letters are usually used to denote elements of sets.

John Venn (1834–1923). John Venn was born into a London suburban family noted for its philanthropy. He attended London schools and got his mathematics degree from Caius College, Cambridge, in 1857. He was elected a fellow of this college and held his fellowship there until his death. He took holy orders in 1859 and, after a brief stint of religious work, returned to Cambridge, where he developed programs in the moral sciences. Besides his mathematical work, Venn had an interest in history and wrote extensively about his college and family.

Venn’s book Symbolic Logic clarifies ideas originally presented by Boole. In this book, Venn presents a systematic development of a method that uses geometric figures, known now as Venn diagrams. Today these diagrams are primarily used to analyze logical arguments and to illustrate relationships between sets. In addition to his work on symbolic logic, Venn made contributions to probability theory described in his widely used textbook on that subject.
There is a special set that has no elements. This set is called the empty set, or null set, and is denoted by $\emptyset$. The empty set can also be denoted by $\{}$ (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

**Definition 3.** The set $A$ is said to be a subset of $B$ if and only if every element of $A$ is also an element of $B$. We use the notation $A \subseteq B$ to indicate that $A$ is a subset of the set $B$.

We see that $A \subseteq B$ if and only if the quantification

$$\forall x (x \in A \rightarrow x \in B)$$

is true. For instance, the set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10. The set of all computer science majors at your school is a subset of the set of all students at your school.

The null set is a subset of every set, that is,

$$\emptyset \subseteq S$$

whenever $S$ is a set. To establish that the null set is a subset of $S$, we must show that every element of the null set is also in $S$. In other words, we must show that the implication "if $x \in \emptyset$, then $x \in S$" is always true. We need only note that the hypothesis of this implication—namely, "$x \in \emptyset$"—is always false to see that this implication is always true. Hence, the empty set is a subset of every set. Furthermore, note that every set is a subset of itself (the reader should verify this). Consequently, if $P$ is a set, we know that $\emptyset \subseteq P$ and $P \subseteq P$.

When we wish to emphasize that a set $A$ is a subset of the set $B$ but that $A \neq B$, we write $A \subset B$ and say that $A$ is a proper subset of $B$. Venn diagrams can be used to show that a set $A$ is a subset of a set $B$. We draw the universal set $U$ as a rectangle. Within this rectangle we draw a circle for $B$. Since $A$ is a subset of $B$, we draw the circle for $A$ within the circle for $B$. This relationship is shown in Figure 2.

One way to show that two sets have the same elements is to show that each set is a subset of the other. In other words, we can show that if $A$ and $B$ are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. This turns out to be a useful way to show that two sets are equal.

Sets may have other sets as members. For instance, we have the sets

$$\{\emptyset, \{a\}, \{a, b\}\} \quad \text{and} \quad \{x \mid x \text{ is a subset of the set }\{a, b\}\}.$$ 

Note that these two sets are equal.
Sets are used extensively in counting problems, and for such applications we need to discuss the size of sets.

**Definition 4.** Let $S$ be a set. If there are exactly $n$ distinct elements in $S$ where $n$ is a nonnegative integer, we say that $S$ is a **finite set** and that $n$ is the **cardinality** of $S$. The cardinality of $S$ is denoted by $|S|$.

**Example 7** Let $A$ be the set of odd positive integers less than 10. Then $|A| = 5$.

**Example 8** Let $S$ be the set of letters in the English alphabet. Then $|S| = 26$.

**Example 9** Since the null set has no elements, it follows that $|\emptyset| = 0$.

We will also be interested in sets that are **not** finite.

**Definition 5.** A set is said to be **infinite** if it is not finite.

**Example 10** The set of positive integers is infinite.

The cardinality of infinite sets will be discussed in Section 1.7. In that section, we will discuss what it means for a set to be countable and show that certain sets are countable while others are not.

**The Power Set**

Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set $S$, we build a new set that has as its members all the subsets of $S$.

**Definition 6.** Given a set $S$, the **power set** of $S$ is the set of all subsets of the set $S$. The power set of $S$ is denoted by $P(S)$.

**Example 11** What is the power set of the set $\{0, 1, 2\}$?

*Solution:* The power set $P(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$  

Note that the empty set and the set itself are members of this set of subsets.
EXAMPLE 12

What is the power set of the empty set? What is the power set of the set \( \{\emptyset\} \)?

Solution: The empty set has exactly one subset, namely, itself. Consequently,
\[
P(\emptyset) = \{\emptyset\}.
\]
The set \( \{\emptyset\} \) has exactly two subsets, namely, \( \emptyset \) and the set \( \{\emptyset\} \) itself. Therefore,
\[
P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.
\]

If a set has \( n \) elements, then its power set has \( 2^n \) elements. We will demonstrate this fact in several ways in subsequent sections of the text.

CARTESIAN PRODUCTS

The order of elements in a collection is often important. Since sets are unordered, a different structure is needed to represent ordered collections. This is provided by \textit{ordered \( n \)-tuples}.

\textbf{DEFINITION 7.} The \textit{ordered \( n \)-tuple} \((a_1, a_2, \ldots, a_n)\) is the ordered collection that has \( a_1 \) as its first element, \( a_2 \) as its second element, \ldots, and \( a_n \) as its \( n \)th element.

We say that two ordered \( n \)-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, \((a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)\) if and only if \( a_i = b_i \), for \( i = 1, 2, \ldots, n \). In particular, 2-tuples are called \textit{ordered pairs}. The ordered pairs \((a, b)\) and \((c, d)\) are equal if and only if \( a = c \) and \( b = d \). Note that \((a, b)\) and \((b, a)\) are not equal unless \( a = b \).

Many of the discrete structures we will study in later chapters are based on the notion of the \textit{Cartesian product} of sets (named after René Descartes). We first define the Cartesian product of two sets.

\begin{quote}
René Descartes (1596–1650). René Descartes was born into a noble family near Tours, France, about 200 miles southwest of Paris. He was the third child of his father’s first wife; she died several days after his birth. Because of René’s poor health, his father, a provincial judge, let his son’s formal lessons slide until, at the age of 8, René entered the Jesuit college at La Fleche. The rector of the school took a liking to him and permitted him to stay in bed until late in the morning because of his frail health. From then on, Descartes spent his mornings in bed; he considered these times his most productive hours for thinking.

Descartes left school in 1612, moving to Paris, where he spent 2 years studying mathematics. He earned a law degree in 1616 from the University of Poitiers. At 18 Descartes became disgusted with studying and decided to see the world. He moved to Paris and became a successful gambler. However, he grew tired of bawdy living and moved to the suburb of Saint-Germain, where he devoted himself to mathematical study.

When his gambling friends found him, he decided to leave France and undertake a military career. However, he never did any fighting. One day, while escaping the cold in an overheated room at a military encampment, he had several feverish dreams, which revealed his future career as a mathematician and philosopher.

After ending his military career, he traveled throughout Europe. He then spent several years in Paris, where he studied mathematics and philosophy and constructed optical instruments. Descartes decided to move to Holland, where he spent 20 years wandering around the country, accomplishing his most important work. During this time he wrote several books, including the \textit{Discours}, which contains his contributions to analytic geometry, for which he is best known. He also made fundamental contributions to philosophy.

In 1649 Descartes was invited by Queen Christina to visit her court in Sweden to tutor her in philosophy. Although he was reluctant to live in what he called “the land of bears amongst rocks and ice,” he finally accepted the invitation and moved to Sweden. Unfortunately, the winter of 1649–1650 was extremely bitter. Descartes caught pneumonia and died in mid-February.
\end{quote}
**Definition 8.** Let $A$ and $B$ be sets. The **Cartesian product** of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(a, b)$ where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}.$$

**Example 13**

Let $A$ represent the set of all students at a university, and let $B$ represent the set of all courses offered at the university. What is the Cartesian product $A \times B$?

*Solution:* The Cartesian product $A \times B$ consists of all the ordered pairs of the form $(a, b)$, where $a$ is a student at the university and $b$ is a course offered at the university. The set $A \times B$ can be used to represent all possible enrollments of students in courses at the university.

**Example 14**

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

*Solution:* The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

The Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or unless $A = B$ (see Exercise 24, at the end of this section). This is illustrated in the following example.

**Example 15**

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where $A$ and $B$ are as in Example 14.

*Solution:* The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

This is not equal to $A \times B$, which was found in Example 14.

The Cartesian product of more than two sets can also be defined.

**Definition 9.** The **Cartesian product** of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i$ belongs to $A_i$ for $i = 1, 2, \ldots, n$. In other words

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \quad \text{for} \quad i = 1, 2, \ldots, n\}.$$

**Example 16**

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

*Solution:* The Cartesian product $A \times B \times C$ consists of all ordered triples $(a, b, c)$, where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.$$
Exercises

1. List the members of the following sets.
   a) \( \{ x \mid x \text{ is a real number such that } x^2 = 1 \} \)
   b) \( \{ x \mid x \text{ is a positive integer less than } 12 \} \)
   c) \( \{ x \mid x \text{ is the square of an integer and } x < 100 \} \)
   d) \( \{ x \mid x \text{ is an integer such that } x^2 = 2 \} \)

2. Use set builder notation to give a description of each of the following sets.
   a) \( \{0, 3, 6, 9, 12\} \)
   b) \( \{-3, -2, -1, 0, 1, 2, 3\} \)
   c) \( \{m, n, o, p\} \)

3. Determine whether each of the following pairs of sets is equal.
   a) \( \{1, 3, 5, 7, 9, 11\}, \{1, 3\} \)
   b) \( \{1\}, \{1\} \)
   c) \( \emptyset, \{\emptyset\} \)

4. Suppose that \( A = \{2, 4, 6\}, B = \{2, 6\}, C = \{4, 6\}, \text{ and } D = \{4, 6, 8\} \). Determine which of these sets are subsets of which other of these sets.

5. For each of the following sets, determine whether 2 is an element of that set.
   a) \( \{x \in \mathbb{R} \mid x \text{ is an integer greater than } 1\} \)
   b) \( \{x \in \mathbb{R} \mid x \text{ is the square of an integer}\} \)
   c) \( \{2, 2\} \)
   d) \( \emptyset, \{\emptyset\} \)
   e) \( \{2, \{2\}\} \)
   f) \( \{\{2\}\} \)

6. For each of the sets in Exercise 5, determine whether \( 2 \) is an element of that set.

7. Determine whether each of the following statements is true or false.
   a) \( x \in \{x\} \)
   b) \( \{x\} \subseteq \{x\} \)
   c) \( \{x\} \subseteq \{\{x\}\} \)
   d) \( \{x\} \subseteq \{\{x\}\} \)
   e) \( \emptyset \subseteq \{x\} \)
   f) \( \emptyset \subseteq \{\{x\}\} \)

8. Use a Venn diagram to illustrate the relationship \( A \subseteq B \text{ and } B \subseteq C \).

9. Suppose that \( A, B, \text{ and } C \) are sets such that \( A \subseteq B \text{ and } B \subseteq C \). Show that \( A \subseteq C \).

10. Find two sets \( A \text{ and } B \) such that \( A \subseteq B \text{ and } A \subseteq B \).

11. What is the cardinality of each of the following sets?
   a) \( \{a\} \)
   b) \( \{a\}, \{a\} \)
   c) \( \{a\}, \{a\} \)
   d) \( \{a\}, \{a\}, \{a\} \)

12. What is the cardinality of each of the following sets?
   a) \( \emptyset \)
   b) \( \emptyset \)
   c) \( \{\emptyset\}, \{\emptyset\} \)
   d) \( \{\emptyset\}, \{\emptyset\}, \{\emptyset\} \)

13. Find the power set of each of the following sets.
   a) \( \{a\} \)
   b) \( \{a\} \)
   c) \( \{\emptyset\}, \{\emptyset\} \)

14. Can you conclude that \( A = B \) if \( A \text{ and } B \) are two sets with the same power set?

15. How many elements does each of the following sets have?
   a) \( P(a, b, \{a, b\}) \)
   b) \( P(\emptyset, a, \{a\}, \{\{a\}\}) \)
   c) \( P(P(\emptyset)) \)

16. Determine whether each of the following sets is the power set of a set.
   a) \( \emptyset \)
   b) \( \emptyset, \{a\} \)
   c) \( \emptyset, \{a\}, \{\{a\}\} \)
   d) \( \emptyset, \{a\}, \{b\}, \{a, b\} \)

17. Let \( A = \{a, b, c, d\} \) and \( B = \{y, z\} \). Find \( A \times B \).

18. What is the Cartesian product \( A \times B \), where \( A \) is the set of courses offered by the mathematics department at a university and \( B \) is the set of mathematics professors at this university?

19. What is the Cartesian product \( A \times B \times C \), where \( A \) is the set of all airlines and \( B \text{ and } C \) are both the set of all cities in the United States?

20. Suppose that \( A \times B = \emptyset \), where \( A \text{ and } B \) are sets. What can you conclude?

21. Let \( A \) be a set. Show that \( \emptyset \times A = A \times \emptyset = \emptyset \).

22. Let \( A = \{a, b, c\}, B = \{x, y\}, \text{ and } C = \{0, 1\} \). Find \( A \times B \times C \).

23. How many different elements does \( A \times B \) have if \( A \) has \( m \) elements and \( B \) has \( n \) elements?

24. Show that \( A \times B \neq B \times A \), when \( A \text{ and } B \) are nonempty unless \( A = B \).

25. Show that the ordered pair \( (a, b) \) can be defined in terms of sets as \( \{\{a\}, \{a, b\}\} \). (Hint: First show that \( \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \) if and only if \( a = c \) and \( b = d \)).

26. In this exercise Russell's paradox is presented. Let \( S \) be the set that contains a set \( x \) if the set \( x \) does not belong to itself, so that \( S = \{x \mid x \notin x\} \).

   a) Show that the assumption that \( S \) is a member of \( S \) leads to a contradiction.
   b) Show that the assumption that \( S \) is not a member of \( S \) leads to a contradiction.

From parts (a) and (b) it follows that the set \( S \) cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

27. Describe a procedure for listing all the subsets of a finite set.