There are 15 problems on 2 pages of this homework assignment.

1. Let $G$ be a group with identity $e$ and $H = \{a \in G \mid a^2 = e\}$.
   (a) Prove that if $G$ is abelian, then $H$ is a subgroup of $G$.
   (b) Is the statement true if $G$ is nonabelian? Justify your answer.

2. Let $(G, \ast)$ be a group and $H$ a nonempty subset of $G$ that is closed under $\ast$.
   (a) Prove that if $G$ is finite, then $H$ is a subgroup of $G$.
   (b) Is the statement true if $G$ is infinite? Justify your answer.

3. If $G$ is cyclic, prove that every subgroup of $G$ is cyclic.

4. A subgroup is proper if it is neither the identity nor the entire group. Prove that if $G$ has no proper subgroups, then $G$ is cyclic. Do not assume that $G$ is finite.

5. Let $G$ be a group and $H$ a nonempty subset of $G$ such that $ab^{-1} \in H$ for all $a, b \in H$. Prove that $H$ is a subgroup of $G$.

6. Let $G$ be a group and $a \in G$. The set $C(a) = \{g \in G \mid ag = ga\}$ is called the centralizer of $a$. Prove that $C(a)$ is a subgroup of $G$ for all $a \in G$.

7. Let $S$ be an infinite set and let $M \subseteq A(S)$ be the set of all $f \in A(S)$ such that $f(s) \neq s$ for at most a finite number of $s \in S$.
   (a) Prove that $M$ is a subgroup of $A(S)$.
   (b) Let $f \in A(S)$. Prove that $f^{-1}Mf = \{f^{-1}gf \mid g \in M\}$ equals $M$.

8. Let $S$ be a nonempty set with $s_1, s_2 \in S$ and $s_1 \neq s_2$. Let $H = \{f \in A(S) \mid f(s_1) = s_1\}$ and $K = \{f \in A(S) \mid f(s_2) = s_2\}$
   (a) Prove that $H$ is a subgroup of $A(S)$.
   (b) Is $K$ a subgroup of $A(S)$? Justify your answer.
   (c) Prove that there is an $g \in A(S)$ such that $g(s_1) = s_2$.
   (d) Prove that if $f \in K$, then $g^{-1}fg \in H$.
   (e) Prove that if $h \in H$, then there exists some $f \in K$ such that $h = g^{-1}fg$.

9. If $m < n$, prove that there is a 1-1 mapping $F : S_m \to S_n$ such that $F(fg) = F(f)F(g)$ for all $f, g \in S_m$.

10. Let $(G, \ast)$ and $(H, \circ)$ be groups with identities $e \in G$, $f \in H$ and $\phi : G \to H$ an isomorphism.
   (a) Prove that $\phi(e) = f$.
   (b) Prove that $\phi(g^{-1}) = (\phi(g))^{-1}$ for all $g \in G$. 
11. Let $G$ be a group with $a \in G$. Prove that the function $\phi : G \to G$ defined by $\phi(g) = aga^{-1}$ for all $g \in G$ is an isomorphism.

12. Let $(G, \ast)$ and $(H, \circ)$ be two groups. Prove that if $\phi : G \to H$ is an isomorphism, then the inverse function $\phi^{-1}$ is an isomorphism from $H$ to $G$.

13. Let $G, H,$ and $K$ be three groups. Prove that if $\phi : G \to H$ and $\psi : H \to K$ are isomorphisms, then the composition $\psi \circ \phi : G \to K$ is an isomorphism.

14. Let $\mathcal{G}$ be the set of all groups. Define a relation on $\mathcal{G}$ where $G \simeq G'$ if and only if $G$ is isomorphic to $G'$ for all $G, G' \in \mathcal{G}$. Prove that this is an equivalence relation.

15. Let $(G, \ast)$ be a group. Define a binary operation $\circ$ on $G$ by $a \circ b = b \ast a$.

   (a) Prove that $(G, \circ)$ is a group.

   (b) Prove that $(G, \ast)$ and $(G, \circ)$ are isomorphic.