There are 19 problems on 2 pages of this assignment.

Let \((s_n)\) be a sequence of real numbers. Then \((s_n)\) is increasing if \(s_n \leq s_{n+1}\), or strictly increasing if \(s_n < s_{n+1}\), for all \(n\). It is decreasing if \(s_n \geq s_{n+1}\), or strictly decreasing if \(s_n > s_{n+1}\), for all \(n\). It is monotonic (or monotone) if it is either increasing or decreasing.

Let \(n_1, n_2, n_3, \ldots\), be a strictly increasing sequence of positive integers \(s\). Then the sequence \((t_k)_{k \in \mathbb{N}}\) where \(t_k = s_{n_k}\) is called a subsequence of the sequence \((s_n)\). If \((s_{n_k})\) converges, or diverges to \(+\infty\) or to \(-\infty\), then this limit is called a subsequential limit of \((s_n)\).

We define

\[
\limsup s_n = \lim_{N \to \infty} \sup \{s_n \mid n > N\}
\]

and

\[
\liminf s_n = \lim_{N \to \infty} \inf \{s_n \mid n > N\}.
\]

If \(s_n\) is not bounded above, \(\sup \{s_n \mid n > N\} = +\infty\) for all \(N\) and we define \(\limsup s_n = +\infty\). Similarly, if \(s_n\) is not below above, \(\inf \{s_n \mid n > N\} = -\infty\) for all \(N\) and we define \(\liminf s_n = -\infty\).

The sequence \((s_n)\) is a Cauchy sequence if for every \(\epsilon > 0\) there exists a number \(N\) such that \(m, n > N\) implies \(|s_n - s_m| < \epsilon\).

1. Let \(s_1 = 1\) and for \(n \geq 1\) let \(s_{n+1} = \sqrt{s_n + 1}\). Assume that \(s_n\) converges and find the limit.

2. Let \(x_1 = 1\) and \(x_{n+1} = 3x_n^2\) for \(n \geq 1\).
   
   (a) Show that if \(a = \lim x_n\) for \(a \in \mathbb{R}\), then \(a = 1/3\) or \(a = 0\).
   
   (b) Does \(\lim x_n\) exist? Explain.
   
   (c) Resolve this apparent contradiction.

3. Consider the sequences for \(n \in \mathbb{N}\) defined as follows:

   \[
a_n = (-1)^n, \quad b_n = \frac{1}{n}, \quad c_n = n^2, \quad d_n = \frac{6n + 4}{7n - 3}.
\]

   (a) For each sequence, give an example of a monotone subsequence.
   
   (b) For each sequence, give its set of subsequential limits.
   
   (c) For each sequence, give its \(\limsup\) and \(\liminf\).
   
   (d) Which of the sequences converges? diverges to \(+\infty\)? diverges to \(-\infty\)?
   
   (e) Which of the sequences is bounded?

4. Repeat problem 3 for the sequences:

   \[
s_n = \cos(n\pi/3), \quad t_n = \frac{3}{(4n + 1)}, \quad u_n = \left(-\frac{1}{2}\right)^n, \quad v_n = (-1)^n + \frac{1}{n}.
\]

5. Repeat problem 3 for the sequences:

   \[
w_n = (-2)^n, \quad x_n = 5(-1)^n, \quad y_n = 1 + (-1)^n, \quad z_n = n \cos(n\pi/4).
\]
6. (a) Let \((s_n)_{n \in \mathbb{N}}\) be a sequence with a subsequence \((s_{n_k})\). Prove by induction that \(n_k \geq k\) for all \(k \in \mathbb{N}\).

(b) Prove that if the sequence \((s_n)\) converges, then every subsequence converges to the same limit.

7. If \((s_n)\) is a sequence of positive real numbers, then \(\lim s_n = +\infty\) if and only if \(\lim (1/s_n) = 0\).

8. Let \((s_n)\) and \((t_n)\) be sequences such that \(\lim s_n = +\infty\) and \(\lim t_n > 0\). (Note that \((t_n)\) could diverge to \(+\infty\) also.) Then \(\lim s_n t_n = +\infty\). \(\text{Hint: do you know or can you prove that there exists a number } m \text{ such that } 0 < m < \lim t_n?\)

9. Suppose that there exists \(N_0\) such that \(s_n \leq t_n\) for all \(n > N_0\).

(a) Prove that if \(\lim s_n = +\infty\), then \(\lim t_n = +\infty\).

(b) Prove that if \(\lim t_n = -\infty\), then \(\lim s_n = -\infty\).

(c) Prove that if \(\lim s_n\) and \(\lim t_n\) exist, then \(\lim s_n \leq \lim t_n\).

10. (a) Show that if \(\lim s_n = +\infty\) and \(k > 0\), then \(\lim (ks_n) = +\infty\).

(b) Show that \(\lim s_n = +\infty\) if and only if \(\lim (-s_n) = -\infty\).

(c) Show that if \(\lim s_n = +\infty\) and \(k < 0\), then \(\lim (ks_n) = -\infty\).

11. (a) Prove if \((s_n)\) is bounded and increasing, then it converges. \(\text{Hint: Does } \sup \{s_n \mid n \in \mathbb{N}\} \text{ exist? Can you prove that } \lim s_n \text{ is the least upper bound?}\)

(b) Prove that all bounded monotone sequences converge.

12. (a) Prove that if \((s_n)\) is an unbounded increasing sequence, then \(\lim s_n = +\infty\).

(b) Prove that if \((s_n)\) is an unbounded decreasing sequence, then \(\lim s_n = -\infty\).

(c) Prove that if \((s_n)\) is monotone, then \(\lim s_n\) exists (as a real number, or \(+\infty\), or \(-\infty\)).

13. Let \(S\) be a bounded nonempty subset of \(\mathbb{R}\) and suppose \(\sup S / \not\in S\). Prove that there is a nondecreasing sequence \((s_n)\) of points in \(S\) such that \(\lim s_n = \sup S\).

14. (a) Prove that the sequence \((s_n)\) converges where \(s_1 = 1\) and \(s_{n+1} = \sqrt{s_n + 1}\).

(b) Prove that the sequence \((x_n)\) diverges to \(+\infty\) where \(x_1 = 1\) and \(x_{n+1} = 3x_n^2\).

15. Let \(s_1 = 1\) and \(s_n = (s_{n+1})/3\) for \(n \geq 1\).

(a) Use induction to show that \(s_n > 1/2\) for all \(n\).

(b) Show that \((s_n)\) is decreasing.

(c) Prove that \(\lim s_n\) exists and find it.

16. Prove that a convergent sequence of real numbers is a Cauchy sequence.

17. Prove that the sequence \(s_n = (-1)^{n+1}(\frac{n}{n+1})\) is not a Cauchy sequence. Conclude it is divergent.

18. Prove that a real-valued Cauchy sequence is bounded.

19. There is a theorem which states that every real-valued Cauchy sequence converges to a real number. Is it true that every rational-valued Cauchy sequence converges to a rational number? Justify your answer.