Exercises

1. Which of the following are posets?
   a) $\mathbb{Z}$
   b) $\mathbb{Z}^* = \{x < y \mid x, y \in \mathbb{Z} \}$
   c) $\mathbb{Z}$
   d) $\mathbb{Z}$

2. Determine whether the relations represented by the following zero–one matrices are partial orders.
   a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
   c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

In Exercises 3–5 determine whether the relation with the directed graph shown is a partial order.

3.\[\text{\includegraphics[width=0.5\textwidth]{image3}}\]
4.\[\text{\includegraphics[width=0.5\textwidth]{image4}}\]
5.\[\text{\includegraphics[width=0.5\textwidth]{image5}}\]

6. Let $(S, R)$ be a poset. Show that $(S, R^{-1})$ is also a poset, where $R^{-1}$ is the inverse of $R$. The poset $(S, R^{-1})$ is called the dual of $(S, R)$.

7. Find the duals of the following posets.
   a) $\{0, 1, 2\}$
   b) $\{0, 1, 2, 3\}$
   c) $\mathbb{Z}$
   d) $\mathbb{Z}$

8. Which of the following pairs of elements are comparable in the poset $(\mathbb{Z}^+, \leq)$?
   a) 5, 15
   b) 6, 9
   c) 8, 16
   d) 7, 7

9. Find two incomparable elements in the following posets.
   a) $\mathcal{P}(\{0, 1, 2\})$
   b) $\mathcal{P}(\{0, 1, 2, 3, 4\})$

10. Let $S = \{1, 2, 3, 4\}$. With respect to the lexicographic order based on the usual “less than” relation,
    a) Find all pairs in $S \times S$ less than $(2, 3)$.
    b) Find all pairs in $S \times S$ greater than $(3, 1)$.
    c) Draw the Hasse diagram of the poset $(S \times S, \leq)$.

11. Find the lexicographic ordering of the following $n$-tuples.
    a) $(1, 1, 2), (1, 2, 1)$
    b) $(0, 1, 2, 3), (0, 1, 3, 2)$
    c) $(1, 0, 1, 0, 1), (0, 1, 1, 0, 1)$

12. Find the lexicographic ordering of the following strings of lowercase English letters:
    a) quack, quick, quicksilver, quicksand, quacking
    b) open, openen, opera, operand, opened
    c) zoo, zero, zoom, zoology, zoological

13. Find the lexicographic ordering of the bit strings 0, 01, 011, 0101, 01001, and 0101 based on the ordering 0 < 1.

14. Draw the Hasse diagram for the “greater than or equals” relation on $\{0, 1, 2, 3, 4, 5\}$.

15. Draw the Hasse diagram for divisibility on the set
    a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$
    b) $\{1, 2, 3, 5, 7, 11, 13\}$
    c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$
    d) $\{1, 2, 4, 8, 16, 32, 64\}$

16. Draw the Hasse diagram for inclusion on the set $\mathcal{P}(S)$ where $S = \{a, b, c, d\}$.

In Exercises 17–19 list all ordered pairs in the partial ordering with the accompanying Hasse diagram.

17.\[\text{\includegraphics[width=0.5\textwidth]{image17}}\]
18.\[\text{\includegraphics[width=0.5\textwidth]{image18}}\]
19.\[\text{\includegraphics[width=0.5\textwidth]{image19}}\]

Let $(S, \preceq)$ be a poset. We say that an element $y \in S$ covers an element $x \in S$ if $x < y$ and there is no element $z \in S$ such that $x < z < y$. The set of pairs $(x, y)$ such that $y$ covers $x$ is called the covering relation of $(S, \preceq)$.

20. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 12\}$?

21. What is the covering relation of the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set of $\{1, 2, 3, 4, 6, 12\}$?

22. Show that the pair $(x, y)$ belongs to the covering relation of the finite poset $(S, \preceq)$ if and only if $x$ is lower than $y$ and there is an edge joining $x$ and $y$ in the Hasse diagram of this poset.

23. Show that a finite poset can be reconstructed from its covering relation.

24. Answer the following questions for the partial order represented by the following Hasse diagram.
25. Answer the following questions concerning the poset $\{(3, 5, 9, 15, 24, 45), \}$. 
   a) Find the maximal elements.
   b) Find the minimal elements.
   c) Is there a greatest element?
   d) Is there a least element?
   e) Find all upper bounds of \{a, b, c\}.
   f) Find the least upper bound of \{a, b, c\}, if it exists.
   g) Find all lower bounds of \{f, g, h\}.
   h) Find the greatest lower bound of \{f, g, h\}, if it exists.

26. Answer the following questions concerning the poset $\{(2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72), \}$. 
   a) Find the maximal elements.
   b) Find the minimal elements.
   c) Is there a greatest element?
   d) Is there a least element?
   e) Find all upper bounds of \{2, 9\}.
   f) Find the least upper bound of \{2, 9\}, if it exists.
   g) Find all lower bounds of \{60, 72\}.
   h) Find the greatest lower bound of \{60, 72\}, if it exists.

27. Answer the following questions concerning the poset $\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. 
   a) Find the maximal elements.
   b) Find the minimal elements.
   c) Is there a greatest element?
   d) Is there a least element?
   e) Find all upper bounds of \{\{2\}, \{4\}\}.
   f) Find the least upper bound of \{\{2\}, \{4\}\}, if it exists.
   g) Find all lower bounds of \{\{1, 3, 4\}, \{2, 3, 4\}\}.
   h) Find the greatest lower bound of \{\{1, 3, 4\}, \{2, 3, 4\}\}, if it exists.

28. Give a poset that has 
   a) a minimal element but no maximal element.
   b) a maximal element but no minimal element.
   c) neither a maximal nor a minimal element.

29. Show that lexicographic order is a partial ordering on the Cartesian product of two posets.

30. Show that lexicographic order is a partial ordering on the set of strings from a poset.

31. Suppose that $(S, \leq_1)$ and $(T, \leq_2)$ are posets. Show that $(S \times T, \leq)$ is a poset where $(s, t) \leq (u, v)$ if and only if $s \leq_1 u$ and $t \leq_2 v$.

32. a) Show that there is exactly one greatest element of a poset, if such an element exists.
   b) Show that there is exactly one least element of a poset, if such an element exists.

33. a) Show that there is exactly one maximal element in a poset with a greatest element.
   b) Show that there is exactly one minimal element in a poset with a least element.

34. a) Show that the least upper bound of a set in a poset is unique if it exists.
   b) Show that the greatest lower bound of a set in a poset is unique if it exists.

35. Determine whether the posets with the following Hasse diagrams are lattices.

36. Determine whether the following posets are lattices.
   a) $\{(1, 3, 6, 9, 12), \}$
   b) $\{(1, 5, 25, 125), \}$
   c) $(\mathbb{Z}, \leq)$
   d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set $S$

37. Show that every nonempty subset of a lattice has a least upper bound and a greatest lower bound.

38. Show that if the poset $(S, R)$ is a lattice then the dual poset $(S, R^{-1})$ is also a lattice.

39. In a company, the lattice model of information flow is used to control sensitive information with security classes represented by ordered pairs $(A, C)$. Here $A$ is an authority level which may be nonproprietary (0), proprietary (1), restricted (2), or registered (3).
A category $C$ is a subset of the set of all projects (Cheetah, Impala, Puma). (Names of animals are often used as code names for projects in companies.)
a) Is information permitted to flow from (Proprietary, \{Cheetah, Puma\}) into (Restricted, \{Puma\})?
b) Is information permitted to flow from (Restricted, \{Cheetah\}) into (Registered, \{Cheetah, Impala\})?
c) Into which classes is information from (Proprietary, \{Cheetah, Puma\}) permitted to flow?
d) From which classes is information permitted to flow into the security class (Restricted, \{Impala, Puma\})?

40. Show that the set $S$ of security classes $(A, C)$ is a lattice, where $A$ is a positive integer representing an authority class and $C$ is a subset of a finite set of compartments, with $(A_1, C_1) \leq (A_2, C_2)$ if and only if $A_1 \leq A_2$ and $C_1 \subseteq C_2$. [Hint: First show that $(S, \leq)$ is a poset and then show that the least upper bound and greatest lower bound of $(A_1, C_1)$ and $(A_2, C_2)$ are $(\max(A_1, A_2), C_1 \cup C_2)$ and $(\min(A_1, A_2), C_1 \cap C_2)$, respectively.]

41. Show that the set of all partitions of a set $S$ with the relation $P_1 \leq P_2$ if the partition $P_1$ is a refinement of the partition $P_2$: is a lattice. (See the preamble to Exercise 27 of Section 6.5.)

42. Show that every totally ordered set is a lattice.
43. Show that every finite lattice has a least element and a greatest element.
44. Give an example of an infinite lattice with
   a) neither a least nor a greatest element.
   b) a least but not a greatest element.
   c) a greatest but not a least element.
   d) both a least and a greatest element.
45. Verify that $(\mathbb{Z} \times \mathbb{Z}, \leq)$ is a well-ordered set, where $\leq$ is lexicographic order, as claimed in Example 7.
46. Show that a finite nonempty poset has a maximal element.

47. Find a compatible total order for the poset with the Hasse diagram shown in Exercise 24.
48. Find a compatible total order for the divisibility relation on the set \{1, 2, 3, 6, 8, 12, 24, 36\}.
49. Find an order different from that constructed in Example 26 for completing the tasks in the development project.
50. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing these tasks is as shown in the following figure.

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**Key Terms and Results**

**TERMS**

**Binary Relation from A to B:** a subset of $A \times B$

**Relation on A:** a binary relation from A to itself (i.e., a subset of $A \times A$)

$S \circ R$: composite of $R$ and $S$

$R^{-1}$: inverse relation of $R$

$R^n$: $n$th power of $R$

**Reflexive:** a relation $R$ on $A$ is reflexive if $(a, a) \in R$ for all $a \in A$

**Symmetric:** a relation $R$ on $A$ is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$

**Transitive:** a relation $R$ on $A$ is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

**N-ary Relation on $A_1, A_2, \ldots, A_n$:** a subset of $A_1 \times A_2 \times \cdots \times A_n$

**Relational Data Model:** a model for representing databases using $n$-ary relations

**Primary Key:** a domain of an $n$-ary relation such that an $n$-tuple is uniquely determined by its value for the domain