Names:__________________________________________________________

Solutions are to be written on the board.

Let $A$, $B$, $C$, and $D$ be subsets of a universal set $U$.

1. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

2. Prove or disprove.
   
   (a) The equality $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ holds.
   
   (b) If $A \cap B = A \cap C$, then $B = C$.
   
   (c) If $A \cup B = A \cup C$, then $B = C$.
   
   (d) If $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$.
   
   (e) The equality $(A \setminus B) \cap (A \setminus C) = A \setminus (B \cup C)$ holds.
   
   (f) The equality $(A \setminus B) \cup (A \setminus C) = A \setminus (B \cap C)$ holds.
   
   (g) The equality $A \times (B \cup C) = (A \times B) \cup (A \times C)$ holds.
   
   (h) The equality $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ holds.
   
   (i) The equality $(A \cup B) \setminus B = A$ holds.
   
   (j) If $A \subseteq C$ and $B \subseteq D$ and $A \cap B = \emptyset$, then $C$ and $D$ are disjoint.
   
   (k) If $A \cup B \neq \emptyset$, then both $A$ and $B$ are nonempty.
   
   (l) If $A \setminus B = B \setminus A$, then $A \setminus B = \emptyset$.
   
   (m) If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.
   
   (n) The equality $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ holds.
   
   (o) The equality $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ holds.

3. Prove or disprove.
   
   (a) The relation $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ holds.
   
   (b) The relation $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ holds.
   
   (c) The relation $\mathcal{P}(A \setminus B) \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$ holds.
   
   (d) The relation $\mathcal{P}(A) \setminus \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$ holds.
   
   (e) The relation $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ holds.
   
   (f) The relation $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ holds.
   
   (g) The relation $\mathcal{P}(A \setminus B) \subseteq (\mathcal{P}(A) \setminus \mathcal{P}(B)) \cup \{\emptyset\}$ holds.
   
   (h) The relation $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$ holds.
   
   (i) If $A$ and $B$ are disjoint sets, then $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are disjoint.
4. Prove the following directly from the definition. You may assume that sums and products of integers are integers.

(a) The sum of two rational numbers is rational.
(b) The sum of an irrational number and a rational number is irrational.
(c) The product of two rational numbers is rational.
(d) The product of an irrational number and a rational number is irrational.
(e) Let $a$ be an irrational number and $r$ a nonzero rational number. If $s$ is a real number, then either $ar + s$ or $ar - s$ is irrational.

5. Prove the following directly from the definition. You may assume that sums and products of integers are integers. Let $a$, $b$, $c$, $d$, $x$ and $y$ be integers with $a \neq 0$ and $b \neq 0$.

(a) If $a \mid b$ and $b \mid c$, then $a \mid c$.
(b) If $a \mid c$ and $b \mid d$, then $ab \mid cd$.
(c) If $a \mid c$ and $a \mid d$, then $a \mid cx + dy$.
(d) If $a \mid c$, then $a^2 \mid c^2$.
(e) If $a \nmid cd$, then $a \nmid c$ and $a \nmid d$. 