Math 6640, Introduction to Numerical PDE’s
Homework Set 3, Due date: Nov 5.

[1] Analyze the accuracy, stability properties of the Du Fort-Frankel scheme for heat equation with a source term,

\[ u_t = bu_{xx} + f(t, x). \]

[2] Consider the leap-frog scheme for

\[ u_t + au_{xxx} = f \]

given by

\[ \frac{v^m_{n+1} - v^m_{n-1}}{2k} + a\delta^2 \delta_0 v^m_m = f^n_m, \]

where \( \delta_0 \) and \( \delta^2 \) are the central divided difference operators for the first and second order derivatives respectively. Take \( \nu = k/h^3 \) as a constant, find the stability condition.

[3] Find the exact Entropy solution for the Riemann problem of Burgers’ equation with the following initial conditions:

(a) \( u_l = 5 \), \( u_r = 3 \),
(b) \( u_l = 3 \), \( u_r = 5 \).

[4] Compute the initial-boundary value problem for \( u_t = bu_{xx} \) on \( x \in [-1, 1] \) with the initial data given by

\[ u_0(x) = \begin{cases} 1 & \text{if } |x| < 0.5, \\ 0.5 & \text{if } |x| = 0.5, \\ 0 & \text{if } |x| > 0.5. \end{cases} \]

by Crank-Nicolson scheme. Solve it up to \( t = 0.5 \). The boundary data and exact solution are given by

\[ u(t, x) = \frac{1}{2} + 2 \sum_{l=0}^{\infty} (-1)^l \frac{\cos \pi (2l + 1)x}{\pi (2l + 1)} e^{-\pi^2 (2l+1)^2 t}. \]

Design your own set up for the computation and compare the solutions with the exact solution. Will the error in \( L_\infty \) and \( L_2 \) norm decrease as the mesh size decreases?

[5] Solve \( u_t + u_x = 0 \), \( x \in [0, 1] \), \( t \in [0, 1.2] \) with the leapfrog scheme and the initial condition \( u_0(x) = \sin 2\pi x \) and the following boundary conditions:

(a) At \( x = 0 \), \( u = \sin 2\pi t \). At \( x = 1 \), use \( v_{M-1}^n = 2v_{M-2}^n - v_{M-1}^n \).
(b) At \( x = 0 \), \( u = \sin 2\pi t \). At \( x = 1 \), use \( u = 0 \).
(c) At \( x = 0 \), use \( v_{M-1}^n = 2v_{M-2}^n - v_{M-1}^n \). At \( x = 1 \), use \( v_{M-1}^n = v_{M-2}^n \).
(d) At \( x = 0 \), \( u = \sin 2\pi t \). At \( x = 1 \), use \( v_{M-1}^n = v_{M-2}^n \).

Comment on your computational results.