Practice Final Exam for Calculus II, Math 1502, December 10, 2010

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414…. Show your work, otherwise credit cannot be given. Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
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<tr>
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Problems related to Block 1:

I: (15 points) Compute with an error less than $10^{-3}$

$$\int_{2}^{3} e^{\frac{1}{x^2}} dx .$$

II: a) (7 points) Compute the limit

$$\lim_{x \to 0} \frac{e^x - \cos x - \sin x}{x^3}$$

b) (8 points) Does the improper integral

$$\int_{0}^{1} \frac{1}{x^2} e^{\frac{1}{x}} dx$$

exist? If yes, compute it.
III: a) (7 points) Is the series
\[ \sum_{k=0}^{\infty} (-1)^k \frac{(k!)^2}{k^{2k}} \]
convergent? Is it absolutely convergent?

b) (8 points) Find the interval of convergence of the power series
\[ \sum_{k=1}^{\infty} (-1)^k k^{-1+\frac{1}{2}} (x - 2)^k \]

IV: (15 points) Solve the initial value problem
\[ y' - \frac{1}{x^2}y = e^{-\frac{1}{2}}, \quad y(1) = \frac{2}{e}. \]
Problems related to Block 3:

V: (20 points) Consider the system of equations

\[
\begin{align*}
2x + y + z &= b \\
x + y - 2z &= 2 \\
x - y + az &= -1
\end{align*}
\]

Determine all values for \(a\) and \(b\) for which this system has a) non solution, b) exactly one solution, c) infinitely many solutions. In the case b) and c) Compute all the solutions in terms of \(a\) and \(b\).

VI: (15 points) A plane in \(\mathbb{R}^3\) passes through the points

\[
\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}
\]

Give two representations of the plane, one in terms of parametrization and one in terms of an equation.
VII: (20 points) Use the least square method to find the distance of the tip of the vector
\[
\begin{bmatrix}
1 \\
1 \\
1 
\end{bmatrix}
\]
to the plane given by
\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix} + t \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]
Solve the problem in two ways, once using the normal equations and then using the QR factorization.

VIII: (15 points) Consider the matrix
\[
\begin{bmatrix}
2 & 3 & 5 & 6 \\
1 & 0 & 1 & 3 \\
4 & 1 & 5 & 12 \\
2 & 1 & 4 & 7
\end{bmatrix}
\]
Find a basis for $\text{Img}(A)$ and for $\text{Ker}(A)$ as well as for $\text{Img}(A^T)$ and for $\text{Ker}(A^T)$. Try do this with a little computation as possible.
IX: (15 points) Graph the curve given by the equation
\[11x^2 - 6xy + 19y^2 = 10.\]

X: (15 points) Diagonalize the matrices

\[a) \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.\]

\[b) \begin{bmatrix} 6 & 9 \\ 4 & 11 \end{bmatrix}.\]
XI: (20 points) Solve the initial value problem given by the system

\[ x' = 8x + 9y \]
\[ y' = 4x + 13y \]
\[ x(0) = 1, \; y(0) = 2 \]  \hspace{1cm} (0.1)

Use both methods, the superposition principle and the exponential of a matrix.

XII: (15 points) Solve the recursive relation, i.e., find \( a_n \) for arbitrary values of \( n \),

\[ a_{n+1} = 8a_n + 9a_{n-1} \]

with \( a_0 = a_1 = 1 \).