This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414…. Show your work, otherwise credit cannot be given. Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.
I: (15 points) Compute with an error less than $10^{-3}$

$$\int_{2}^{3} e^{\frac{y}{x^2}} dx .$$

$$e^y = \sum_{k=0}^{n} \frac{y^k}{k!} + \frac{c y^{n+1}}{(n+1)!}$$

where $c$ is some number between 0 and $y$. Now we set $y = \frac{1}{x^2}$ and note that since $x$ ranges between 2 and 3 the variable $y$ ranges between 1/4 and 1/9. Hence we know that $c$ can be a number that must be somewhere between 0 and 1/4 and since $e^y$ is monotone increasing we take $c = 1/4$ to obtain an upper bound on the remainder of the form

$$\frac{e^{1/4}y^{n+1}}{(n+1)!} .$$

Now from what we know about the exponential function $e < 3$ and hence $e^{1/4} < 3^{1/4}$ which is some number less than 2. Thus we find that

$$0 < \int_{2}^{3} \left[ e^{\frac{1}{x^2}} - \sum_{k=0}^{n} \frac{x^{-2k}}{k!} \right] dx \leq \frac{2}{(n+1)!} \int_{2}^{3} x^{-2(n+1)} dx$$

$$= \frac{2}{(n+1)!} \frac{1}{2n+1} [2^{-2n-1} - 3^{-2n-1}] < \frac{2}{(n+1)!} \frac{1}{2n+1} 2^{-2n-1} .$$

If we choose $n = 3$ we find that the remainder of the integral is bounded by

$$\frac{2}{4!} \frac{1}{7} 2^{-7} = \frac{1}{84 \times 128} < \frac{1}{1000} .$$

Integrating the sum in the integral yields

$$\sum_{k=0}^{3} \left[ 2^{-2k+1} - 3^{-2k+1} \right] \frac{1}{k!(2k-1)} .$$

II: a) (7 points) Compute the limit

$$\lim_{x \to 0} \frac{e^x - \cos x - \sin x}{x^3}$$
Using the Taylor expansion for all the functions in the numerator yields
\[ e^x - \cos x - \sin x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots - 1 + \frac{x^2}{2} + \cdots - x - \frac{x^3}{3!} + \cdots \]
The leading order coefficient is \( x^2 \) and hence the limit does not exist.

b) (8 points) Does the improper integral
\[ \int_0^1 \frac{1}{x^2} e^{\frac{x}{2}} \, dx \]
exist? If yes, compute it.

We have to compute
\[ \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{x^2} e^{\frac{x}{2}} \, dx \]
and using the substitution
\[ u = \frac{1}{x} \]
this integral can be rewritten as
\[ \int_1^\frac{1}{\varepsilon} e^u \, du = e^{\frac{1}{\varepsilon}} - e^{-1}. \]
Clearly the limit as \( \varepsilon \to 0 \) does not exist.
III: a) (7 points) Is the series
\[ \sum_{k=0}^{\infty} (-1)^k \frac{(k!)^2}{k^{2k}} \]
convergent? Is it absolutely convergent?
Using the ratio test we get that
\[ \frac{((k+1)!)^2 (k^{2k})}{(k+1)^{2k+2} (k!)^2} = \left( \frac{k}{k+1} \right)^{2k} \]
which converges to \( \frac{1}{e^2} < 1 \). Hence the series is absolutely convergent and hence, in particular, convergent.

b) (8 points) Find the interval of convergence of the power series
\[ \sum_{k=1}^{\infty} (-1)^k \frac{k^{-1+\frac{1}{\pi}} (x-2)^k}{k^{1-\frac{1}{\pi}}} \]
Set
\[ a_k = \frac{|x-2|^k}{k^{1-\frac{1}{\pi}}} \]
and note that
\[ \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = |x-2| \]
hence the interval of convergence contains the interval (1, 3). Looking at the end point \( x = 3 \) we find the series
\[ \sum_{k=1}^{\infty} (-1)^k k^{-1+\frac{1}{\pi}} \]
which converges since it is alternating and the coefficients decrease monotonically to zero. Indeed the statement
\[ (k+1)^{-1+\frac{1}{\pi}} < k^{-1+\frac{1}{\pi}} \]
is equivalent to the statement
\[ (k+1)^{1-\frac{1}{\pi}} > k^{1-\frac{1}{\pi}} \]
or
\[ (k+1)^{\frac{k}{k-1}} > k^{\frac{k-1}{k-1}} \]
Since \( k \geq 1 \) and since \( \frac{k}{k+1} > \frac{k-1}{k} \) we have that

\[
(k + 1)^{\frac{k}{k+1}} > (k + 1)^{\frac{k-1}{k}} > k^{\frac{k-1}{k}}.
\]

Hence the series converges at \( x = 3 \). At \( x = 1 \), however, the series diverges, since

\[
k^{-1+\frac{1}{k}} \times k \to 1
\]
as \( k \to \infty \) and since \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges, we also have that

\[
\sum_{k=1}^{\infty} k^{-1+\frac{1}{k}}
\]
diverges, by the limit comparison test.

**IV: (15 points)** Solve the initial value problem

\[
y' - \frac{1}{x^2} y = e^{-\frac{1}{x}}, \quad y(1) = \frac{2}{e}.
\]

Multiply the equation by the integrating factor \( e^{\frac{1}{x}} \)

\[
e^{\frac{1}{x}}y' - \frac{1}{x^2}e^{\frac{1}{x}}y = \left(e^{\frac{1}{x}}y\right)' = 1.
\]

Hence

\[
y(x) = xe^{-\frac{1}{x}} + ce^{-\frac{1}{x}}
\]

where \( c \) is a constant. \( y(1) = \frac{2}{e} \) yields \( c = 1 \) and hence our solutions is

\[
y(x) = (x + 1)e^{-\frac{1}{x}}.
\]
Name:

Section:

Name of TA:

Problems related to Block 3:

V: (20 points) Consider the system of equations

\begin{align*}
2x + y + z &= b \\
 x + y - 2z &= 2 \\
x - y + az &= -1
\end{align*}

Determine all values for \( a \) and \( b \) for which this system has a) non solution, b) exactly one solution, c) infinitely many solutions. In the case b) and c) Compute all the solutions in terms of \( a \) and \( b \). The augmented matrix is

\[
\begin{bmatrix}
2 & 1 & 1 & | & b \\
1 & 1 & -2 & | & 2 \\
1 & -1 & a & | & -1
\end{bmatrix}
\]

Switching the first and second row leads to

\[
\begin{bmatrix}
1 & 1 & -2 & | & 2 \\
2 & 1 & 1 & | & b \\
1 & -1 & a & | & -1
\end{bmatrix}
\]

Row reduction leads to

\[
\begin{bmatrix}
1 & 1 & -2 & | & 2 \\
0 & -1 & 5 & | & b - 4 \\
0 & 0 & a - 8 & | & 5 - 2b
\end{bmatrix}
\]

If \( a = 8 \) and \( 2b \neq 5 \) there is no solution. If \( a \neq 8 \) there is always a unique solutions and if \( a = 8 \) and \( 2b = 5 \) there are infinitely many solutions.

If \( a \neq 8 \) we can use back substitution and obtain:

\[
z = \frac{5 - 2b}{a - 8} , \quad y = \frac{5 - 2b}{a - 8} + 4 - b , \quad x = -3\frac{5 - 2b}{a - 8} + b - 2
\]

If \( a = 8 \) and \( 2b = 5 \) then the row reduced augmented matrix is

\[
\begin{bmatrix}
1 & 1 & -2 & | & 2 \\
0 & -1 & 5 & | & -\frac{3}{2} \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\]
and we find $z = t$, $y = 5t + \frac{3}{2}$ and $x = -3t + \frac{1}{2}$.

**VI:** (15 points) A plane in $\mathbb{R}^3$ passes through the points

\[
\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}
\]

Give two representations of the plane, one in terms of parametrization and one in terms of an equation.

The plane is spanned by the vectors $\mathbf{v}_1 = \mathbf{p}_2 - \mathbf{p}_1$, $\mathbf{v}_2 = \mathbf{p}_3 - \mathbf{p}_1$ and passes through the point $\mathbf{p}_1$. Hence it is given by the parametrization

\[
\mathbf{x}(s,t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

To find the equation we have to find a vector that is perpendicular to both vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and inspection shows that the vector $(1, 0, -1)$ does the job. We have to make sure that the vector $(1, 1, 1)$ has its tip on the plane. Hence the equation is given by

\[x - z = 0 .\]
VII: (20 points) Use the least square method to find the distance of the tip of the vector
\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]
to the plane given by
\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
+ s \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}
+ t \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]
Solve the problem in two ways, once using the normal equations and then using the QR factorization.

If we set
\[
A = \begin{bmatrix}
2 & 1 \\
1 & 1 \\
-1 & 1
\end{bmatrix}
\]

then we have to solve the least square problem
\[A\vec{x} = \vec{b}\]
where
\[
\vec{b} = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}
\]

The normal equations are \(A^TA\vec{x} = A^T\vec{b}\) and hence we have to solve
\[
\begin{bmatrix}
6 & 2 \\
2 & 3
\end{bmatrix}
\vec{x} = \begin{bmatrix}
0 \\
2
\end{bmatrix}
\]
which yields \(x = -2/7\), \(y = 6/7\). Thus, the vector in \(Img(A)\) that is closest to \(\vec{b}\) is given by
\[
A\vec{x} = \frac{2}{7} \begin{bmatrix}
1 \\
2 \\
4
\end{bmatrix}
\]
For the distance we have to calculate the length of
\[
\begin{bmatrix}
0 \\
1 \\
1 \\
\end{bmatrix}
- \frac{2}{7}
\begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}
= \frac{1}{7}
\begin{bmatrix}
-2 \\
3 \\
-1 \\
\end{bmatrix},
\]
given by
\[
\sqrt{\frac{2}{7}}.
\]

Now we use the QR factorization. We have to find an orthonormal basis for \( \text{Img}(A) \).

One vector is
\[
\frac{1}{\sqrt{6}}
\begin{bmatrix}
2 \\
1 \\
-1 \\
\end{bmatrix}
\]
and the other one is then found by looking at the vector
\[
\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}
- \frac{2}{6}
\begin{bmatrix}
2 \\
1 \\
-1 \\
\end{bmatrix}
= \frac{1}{3}
\begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}
\]
which normalized equals
\[
\frac{1}{\sqrt{21}}
\begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}
\]
Hence \( Q \) is given by
\[
Q =
\begin{bmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\
\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\
\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\
\end{bmatrix}
\]
and hence with \( Q^T A = R \) we get
\[
\begin{bmatrix}
\frac{6}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\
0 & \frac{7}{\sqrt{21}} \\
\end{bmatrix}.
\]
Now \( R\vec{x} = Q^T \vec{b} \) leads to \( x = -2/7, y = 6/7 \) which checks. However, in order to compute the shortest distance the matrix \( R \) is not important. The projection of \( \vec{b} \) onto \( \text{Img}(A) \) is given by
\[
QQ^T \vec{b} = Q
\begin{bmatrix}
0 \\
\frac{6}{\sqrt{21}} \\
\end{bmatrix}
= \frac{2}{7}
\begin{bmatrix}
1 \\
2 \\
4 \\
\end{bmatrix}.
which checks with what we had before, and the distance vector from the tip of the vector $\vec{b}$ to $\text{Img}(A)$ is given by

$$\vec{b} - QQ^T\vec{b} = \frac{1}{7} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix},$$

as we had before. Of course the distance is then the same number as before.

**VIII:** (15 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 0 & 1 & 3 \\ 4 & 1 & 5 & 12 \\ 2 & 1 & 4 & 7 \end{bmatrix}$$

Find a basis for $\text{Img}(A)$ and for $\text{Ker}(A)$ as well as for $\text{Img}(A^T)$ and for $\text{Ker}(A^T)$. Try do this with a little computation as possible. Row reducing $A$ yields

$$\begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix},$$

as a basis vector for the kernel of $A$. Thus we can say that the dimension of $\text{Img}(A)$ which is the same as the dimension of $\text{Img}(A^T)$ equals 3. Thus $\text{Ker}(A^T)$ is one dimensional. The row reduced matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\text{Img}(A^T)$ is the orthogonal complement we have to find three linearly independent vectors that are perpendicular to the above vector. Thus we have to solve

$$-2w + x - y + z = 0$$

which leads to the one-one parametrization $z = t$, $y = s$, $x = r$ and $w = \frac{1}{2}[r - s + t]$. Hence we get

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$
as a basis for $\text{Img}(A^T)$. We know that the first three columns of the matrix are pivotal columns and hence we have that the first three vectors from a basis for $\text{Img}(A)$. To find a basis for $\text{Ker}(A^T)$ we have to find a vector perpendicular to the first three column vectors which can be found by row reducing the system

$$
\begin{bmatrix}
2 & 1 & 4 & 2 \\
3 & 0 & 1 & 1 \\
5 & 1 & 5 & 4
\end{bmatrix}
$$

which yields

$$
\begin{bmatrix}
2 & 1 & 4 & 2 \\
0 & 3 & 10 & 4 \\
0 & 0 & 0 & 2
\end{bmatrix}
$$

and from which we get that the basis for the kernel of $A^T$ is

$$
\begin{bmatrix}
1 \\
10 \\
-3 \\
0
\end{bmatrix}
$$
IX: (15 points) Graph the curve given by the equation

\[ 11x^2 - 6xy + 19y^2 = 10. \]

The associated matrix is given by

\[
\begin{bmatrix}
11 & -3 \\
-3 & 19
\end{bmatrix}
\]

whose characteristic polynomial is \( \mu^2 - 30\mu + 20 = (\mu - 10)(\mu - 20) \). Hence the eigenvalues are \( \mu_1 = 10, \mu_2 = 20 \). The associated eigenvectors are

\[
\vec{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\]

and

\[
\vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}.
\]

In the \( u - v \) plane the curve is given by

\[ u^2 + 2v^2 = 1 \]

which is an ellipse whose semiaxis in the \( u \)-direction has length 1 and whose semiaxis in the direction \( v \) has length \( 1/\sqrt{2} \). To get the picture in the \( x - y \) plane we have to rotate the \( u - v \) picture by the rotation matrix

\[
U = [\vec{u}_1, \vec{u}_2] = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}.
\]

X: (15 points) Diagonalize the matrices

\[ a) \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}. \]
The eigenvalue 6 has the eigenvector

\[
\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

Perpendicular to this is the vector

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}
\]

which is an eigenvector with eigenvalue 3. Thus the vector

\[
\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}
\]

which is perpendicular to both must be an eigenvector, since the matrix is symmetric. The associated eigenvalue is 3 also.

\[b) \begin{bmatrix} 6 & 9 \\ 4 & 11 \end{bmatrix}\]

The characteristic polynomial is \(\mu^2 - 17\mu + 30 = (\mu - 15)(\mu - 2)\). The eigenvector associated with \(\mu_1 = 15\) is the vector

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

and the one associated to \(\mu_2 = 2\) is

\[
\begin{bmatrix} 9 \\ -4 \end{bmatrix}
\]
XI: (20 points) Solve the initial value problem given by the system
\[
\begin{align*}
  x' &= 8x + 9y \\
  y' &= 4x + 13y \\
  x(0) &= 1, \quad y(0) = 2
\end{align*}
\] (0.1)

Use both methods, the superposition principle and the exponential of a matrix.

The eigenvalues and the corresponding eigenvectors are

\[
\mu_1 = 17, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

and

\[
\mu_2 = 4, \quad \vec{u}_2 = \begin{bmatrix} 9 \\ -4 \end{bmatrix}
\]

The general solution is then given by

\[
ae^{17t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{4t} \begin{bmatrix} 9 \\ -4 \end{bmatrix}
\]

Using the intial conditions we arrive at

\[
\frac{22}{13} e^{17t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{13} e^{4t} \begin{bmatrix} 9 \\ -4 \end{bmatrix}.
\]
XII: (15 points) Solve the recursive relation, i.e., find \( a_n \) for arbitrary values of \( n \),

\[
a_{n+1} = 8a_n + 9a_{n-1}
\]

with \( a_0 = a_1 = 1 \).

Writing

\[
\vec{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}
\]

we can write the recursion as

\[
\vec{x}_{n+1} = A\vec{x}_n
\]

where

\[
A = \begin{bmatrix} 8 & 9 \\ 1 & 0 \end{bmatrix}.
\]

The solution can be gotten via

\[
\vec{x}_n = A^{n-1}\vec{x}_1.
\]

\( A \) is diagonalized by

\[
A = \begin{bmatrix} 9 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & 1 \\ -1 & 9 \end{bmatrix}.
\]

Hence

\[
A^{n-1} = \begin{bmatrix} 9 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9^{n-1} & 0 \\ 0 & (-1)^{n-1} \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & 1 \\ -1 & 9 \end{bmatrix}.
\]

Applying this to the initial condition \( \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) we find that

\[
a_n = \frac{1}{5} (9^n + 4(-1)^n).
\]