Topics for Test 4

You should be familiar with the following concepts:

**Subspace** $S$ of $\mathbb{R}^n$, which is a subset of $\mathbb{R}^n$ with the property that with any two vectors $\vec{x}, \vec{y} \in S$, $\vec{x} + \vec{y} \in S$ and for all $a \in \mathbb{R}$ and all $\vec{x} \in S$, $a\vec{x} \in S$. Important examples are the kernel of an $m \times n$ matrix $A$, i.e., $\text{Ker}(A) \subset \mathbb{R}^n$ and $\text{Img}(A) \subset \mathbb{R}^m$, the image of an $m \times n$ matrix $A$.

A **spanning set** of a subspace $S \subset \mathbb{R}^n$, which is a collection of vectors so that every vector in $S$ can be written as a linear combination of them.

A collection of vectors is **linearly independent** if no vector of this collection can be written as a linear combination of the others. Alternatively, this means that the matrix $A$ which has those vectors as columns has a kernel $\text{Ker}(A)$ that consists only of the zero vector.

A **basis** of a subspace $S$ is a collection of vectors that spans $S$ and is linearly independent. Every basis of the subspace $S$ has the same number of vectors and this number is called the **dimension** of $S$.

For an $m \times n$ matrix $A$ there is the important dimension formula

$$\dim(\text{Ker}(A)) + \dim(\text{Img}(A)) = n$$

If $S$ is a subspace of $\mathbb{R}^n$ then the **orthogonal complement** of $S$, which is denoted by $S^\perp$ consists of all vectors that are perpendicular to every vector in $S$. The important theorem here is that

$$[S^\perp]^\perp = S$$

If $A$ is an $m \times n$ matrix then

$$\text{Ker}(A) \oplus \text{Img}(A^T) = \mathbb{R}^n$$

$$\text{Ker}(A^T) \oplus \text{Img}(A) = \mathbb{R}^m$$
The meaning of these formulas is that

\[ \text{Ker}(A)\perp = \text{Img}(A^T) \]

both are subspaces of \( \mathbb{R}^n \). Likewise,

\[ \text{Img}(A)\perp = \text{Ker}(A^T) . \]

An \( n \times n \) matrix whose kernel consists only of the zero vector is invertible.

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The above concepts have a computational side to them.

Row reduction leads you to see the pivotal columns and the non-pivotal columns. For an \( m \times n \) matrix \( A \), the pivotal columns are a basis for \( \text{Img}(A) \). The number \( r(A) \) of those columns, is called the **rank of the matrix** \( A \), which equals to the dimension of the image of \( A \), i.e.,

\[ \text{dim}(\text{Img}(A)) = r(A) . \]

The number of non-pivotal columns determines the number of free variables which is the same as \( \text{dim}(\text{Ker}(A)) \).

You can check whether the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) are linearly independent by computing the kernel of the matrix \( A = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k] \). If the kernel consists only of the zero vector, then the vectors are linearly independent. So, row reduction is important!

Very important are the least square problems. The **normal equation**

\[ A^T A \vec{x} = A^T \vec{b} \]

has always a solution, which in general is not unique. If \( \vec{x}^* \) denotes the solution, then

\[ A \vec{x}^* \]
is the vector in $\text{Img}(A)$ that is closest to the vector $\vec{B}$.

This leads to the **projection onto** $\text{Img}(A)$,

$$P = A(A^TA)^{-1}A^T$$

A nicer way of computing such projections as the Gram-Schmidt procedure, which allows from a spanning set $\vec{v}_1, \ldots, \vec{v}_\ell$ to obtain an **orthonormal basis** $\vec{u}_1, \ldots, \vec{u}_k$ where $k \leq \ell$. Note that $k = \ell$ if the $v$-vectors form a basis.

The matrix

$$Q = [\vec{u}_1, \ldots, \vec{u}_k]$$

is an isometry and the matrix $A = [\vec{v}_1, \ldots, \vec{v}_\ell]$ can be written as

$$A = QR$$

the **QR factorization** where $R$ is an upper triangular matrix. We have that

$$R = Q^TA.$$

If a subspace $S$ is spanned by $\vec{v}_1, \ldots, \vec{v}_\ell$ then

$$QQ^T$$

is the orthogonal projection onto $\text{Img}(A)$.

Least square problems can be elegantly solved once the **QR** factorization is available. The equation

$$A\vec{x} = QQ^T\vec{b}$$

has always a solution, since $QQ^T\vec{b} \in \text{Img}(A)$. Hence

$$Q^TA\vec{x} = R\vec{x} = Q^T\vec{b}$$

and $R$ is already in row reduced form.