15) Given: \( E \) is a metric space \( \mathbb{C} \subseteq E \)

\( p \in S \) is an interior point

Interior point: \( \exists \ v \in S \) such that \( B(p; r) \subseteq S \)

Claim: The set of all interior points of \( S \)

is an open subset of \( E \) that contains

all other open subsets of \( E \) that are contained in \( S \).

Proof: Let \( A \) be an open subset of \( E \) that

is contained in \( S \) and \( B(p; r) \) be an

open ball in \( E \) of radius \( r \).

\[ A \subseteq S \quad B(p; r) = \{ q \in E : d(p; q) < r \} \]

Consider \( S_i \) to be the set of all

interior points in \( S \).

\[ S_i = \{ v \in S : \exists \ r > 0 \text{ s.t. } B(p; r) \subseteq S \} \]

Since \( A \) is open, \( A = \{ v \in A : \exists \ u \in A \text{ s.t. } B(p; r) \subseteq A \} \)

Since \( B(p; r) \subseteq A \) and \( A \subseteq S \),

it follows that \( B(p; r) \subseteq A \cap S \) and

\( B(p; r) \subseteq S \). It also follows that

\( A \subseteq S \).

Let \( q, s \in B(p; r) \) and \( \epsilon = \frac{r}{2}, \epsilon > 0 \).
Take \( g, s \in B(p, r) \) and let \( r = \frac{\varepsilon}{2}, \varepsilon > 0 \).
For \( p \in S^i \), let \( B(p, r) \subset S \) where \( r = \varepsilon, \varepsilon > 0 \).
\( \Rightarrow d(g, r) \leq d(g, s) + d(s, r) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \)
Since \( B(g, r) \subset B(p, r) \subset S = B(p, \frac{\varepsilon}{2}) \subset B(p, \varepsilon) \subset \text{AC} \subset S^i \subset \text{CE} \)
g \in S^i \) and \( S^i \) is open.

\( \Rightarrow \) The set of interior points of \( S \) is an open subset of \( E \) that contains all other open subsets of \( E \) that are contained in \( S \).
16) a. Given: E - Metric Space \( \text{SCE} \).
   \( S \) - the closure of \( S \).
   Closure of \( S \) - the intersection of all closed subsets of \( E \) that contain \( S \).

   Claim: \( \overline{S} \supset S \), \( S \) is closed iff \( \overline{S} = S \).

   Proof: Let \( \overline{S} = S_1 \cap S_2 \cap \ldots \cap S_n \). By definition of the closure of \( S \), \( S_1 \cap S_2 \cap \ldots \cap S_n \) are closed subsets of \( E \) that contain \( S \Rightarrow S_1 \cap S_2 \cap \ldots \cap S_n \supset S \). Therefore, \( \overline{S} \supset S \).

   Assume \( S \) is closed and \( \overline{S} \supset S_1 \cap S_2 \cap \ldots \cap S_n \), then since by definition of closure that \( S_1 \cap S_2 \cap \ldots \cap S_n \) is closed, then \( S \) is also closed and \( S = \overline{S} \).

   Assume \( S \) is closed and \( \overline{S} \supset S_1 \cap S_2 \cap \ldots \cap S_n \), then since by definition of closure that \( S_1 \cap S_2 \cap \ldots \cap S_n \) is closed, then \( S \) is also closed and \( \overline{S} \supset S \).

   Since \( S \) is closed by \( \overline{S} \supset S \) and \( \overline{S} \supset S \), then \( S \) is closed iff \( \overline{S} = S \).

   Note: If \( S \) were open, then \( S_1 \cap S_2 \cap \ldots \cap S_n \) would also be open by the definition of closure, \( S \neq S_1 \cap S_2 \cap \ldots \cap S_n \) if for every \( i = 1, 2, \ldots, n \), \( S_i \) was an open set.
\( i=1,2,\ldots, n, \quad S_i \text{ was an open set and } S \subseteq S_1 \cap S_2 \cap \ldots \cap S_n \Rightarrow \overline{S} \) would not be the closure of \( S \).

\[ \Rightarrow \overline{S} \subseteq S \text{ and } S \text{ is closed iff } \overline{S} = S. \]

(2) \text{ Given: } E \text{ - metric space, } S \subseteq E

\[ \overline{S} \text{ - closure of } S \]

\[ \overline{S} \subseteq S \text{ and } S \text{ is closed iff } \overline{S} = S \]

Claim: \( \overline{S} \) is the set of all limits of sequences of points of \( S \) that converge in \( E \).

Proof: Let \( p_n \) be a convergent sequence and \( p_n \to p \). Assume \( p_n \in S \), then by \( S \subseteq \overline{S} \), \( p_n \in \overline{S} \). Since \( \overline{S} \) is closed, then \( p \in \overline{S} \).

Assume \( p \in \overline{S} \), then there exists to exist a \( p_n \in S \) s.t. \( p_n \to p \). Let \( \epsilon > 0 \), \( \exists \) an integer \( N \) s.t. \( |p_n - p| < \epsilon \), pick an integer \( M \) s.t. \( |p_m - p| < \frac{\epsilon}{2} \) for every \( n > N \) and \( n < M < N+1 \) and \( p_m \neq p \). Then, \( p_m \in B(p, \frac{\epsilon}{2}) \cap S \Rightarrow B(p, \frac{\epsilon}{2}) \cap S \neq \emptyset \).

Let \( p_n \in B(p, \epsilon) \cap S \) and \( \epsilon = \frac{1}{n} \). Then, \( |p_n - p| < \frac{1}{n} \Rightarrow p_n \in S \) and \( p_n \to p \).

(3) \text{ Given: } E \text{ - metric space, } S \subseteq E
$\overline{S}$ - closure of $S$
$\overline{S} = \bigwedge_{i=1}^{n} S_i$, for every $S_i, i=1,2,\ldots,n$, $S_i$ is closed
$\overline{S} \supset S$
$S$ is closed iff $\overline{S} = S$
$\overline{S}$ is the set of all limit points of sequences of $S$ that converge in $E$

Claim: A point $p \in E$, $p \in \overline{S}$ iff any ball in $E$ with center $p$ contains points of $S$. This is true iff $p$ is not an interior point of $S^c$.

Proof: This is the same as saying that $\overline{S} = \{ p \in E : d(p, S) \leq 0 \}$. For this to be true, then $\forall \varepsilon > 0$, $B(p, \varepsilon) \cap S \neq \emptyset$ and $B(p, \varepsilon) \cap S^c \neq \emptyset$. $B(p, \varepsilon) \cap S \neq \emptyset$ is true by part (b) when $\varepsilon = \frac{1}{n}$ and $p \in B(p, \frac{1}{n}) \cap S \Rightarrow p \in S$.

For $B(p, \varepsilon) \cap S^c \neq \emptyset$, let $p$ be an interior point of $S^c$. By definition of an interior point, $B(p, \varepsilon) \subset S^c$ which contradicts $B(p, \varepsilon) \cap S \neq \emptyset$, therefore $p \notin$ the interior of $S^c$. Let $p$ be a boundary point of $S^c$, then $B(p, \varepsilon) \cap S^c \neq \emptyset$ and $B(p, \varepsilon) \cap S \neq \emptyset$ as $B(p, \varepsilon)$ would contain at least one point of $S$.

$\Rightarrow$ A point $p \in E$ is in $\overline{S}$ iff a ball in $E$ of center $p$ contains points of $S$, iff
p is not an interior point of \( \mathcal{C} \).
If \( \{a_n\} \) is a bounded sequence of real numbers

\[
\lim \sup \ a_n = \lim a_n = \text{l.u.b.} \{ x \in \mathbb{R} : a_n < x, \text{ for many } n's \}
\]

\[
\lim \ inf \ a_n = \lim a_n = \text{g.l.b.} \{ x \in \mathbb{R} : a_n > x, \text{ for many } n's \}
\]

(a) Prove that \( \lim a_n \leq \lim a_n \).

Set \( \lim a_n = a \) and \( \lim a_n = \bar{a} \). We want to show that \( a \leq \bar{a} \).

Consider \( a + \varepsilon \). There are only finitely many \( n \)'s with \( a_n \geq a + \varepsilon \).

Therefore, all but finitely many \( a_n \)'s satisfy that \( a_n < \bar{a} + \varepsilon \).

Now consider \( a - \varepsilon \). There are only finitely many \( a_n \)'s with \( a_n \leq a - \varepsilon \).

So, all but finitely many \( a_n \)'s satisfy that \( a_n > a - \varepsilon \).

It implies there exist at least one \( a_n \) with \( a - \varepsilon < a_n < \bar{a} + \varepsilon \)

\[
\Rightarrow a - \varepsilon < \bar{a} + \varepsilon
\]

\[
\Rightarrow a < \bar{a} + 2\varepsilon
\]

\[
\Rightarrow a \leq \bar{a} \text{ since } 2\varepsilon > 0 \text{ and arbitrary.}
\]

(b) \( \lim a_n = \lim a_n \) if and only if \( a_n \) converges.

(\( \Rightarrow \)) If \( \lim a_n = \lim a_n = a \), then \( a_n \) converges because all but finitely many \( a_n \)'s satisfy \( a - \varepsilon < a_n < a + \varepsilon \),

where \( \varepsilon > 0 \) and arbitrarily small.

(\( \Leftarrow \)) \( \varepsilon > 0 \). There exists \( N \) such that \( a - \varepsilon < a_n < a + \varepsilon \) for all \( n > N \); where

\[
a - \varepsilon < a_n < a + \varepsilon \text{ for all but finitely many } a_n's. \quad \text{--- 1}
\]

\[
a - \varepsilon < a_n < a + \varepsilon \text{ for all but finitely many } a_n's. \quad \text{--- 2}
\]

There exist \( \infty \) many \( a_n \)'s with \( a_n > \bar{a} - \varepsilon \) because \( \bar{a} \) is the least upper bound of the set

where all the elements of that set is less than \( a_n \). From 2, \( a + \varepsilon > a_n > \bar{a} - \varepsilon \Rightarrow a \geq \bar{a} \), \( \varepsilon > 0 \) arbitrarily small.

There exist \( \infty \) many \( a_n \)'s with \( a_n < a + \varepsilon \) because \( a \) is the greatest lower of the set

where all the elements of that set is greater than \( a_n \). From 1, \( a - \varepsilon < a_n < a + \varepsilon \Rightarrow a < a \), \( \varepsilon > 0 \) arbitrarily small,

\[
\Rightarrow \ a > a \geq a \Rightarrow a = a = a
\]
28 \[ |z| = d(z,0) \]

a) \[ |x+iy| = d(x+iy,0) \text{, where } x \text{ and } y \in \mathbb{R} \]

\[ |x+iy| = d((x,y),(0,0)) = \sqrt{x^2+y^2} \]

- \[ x+iy = (x+iy,0) \]

b) Suppose that \[ z_1 = x_1 + iy_1 \] and \[ z_2 = x_2 + iy_2 \].

\[ z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 + i(y_1 + y_2) \]

\[ |z_1 + z_2| = |x_1 + x_2 + i(y_1 + y_2)| \]

It is known that \[ |z_1| = \sqrt{x_1^2+y_1^2} \] and \[ |z_2| = \sqrt{x_2^2+y_2^2} \], (from the definition of \( |x+iy| = \sqrt{x^2+y^2} \) of complex numbers)

We need to show \[ |z_1 + z_2| = |x_1 + x_2 + i(y_1 + y_2)| \]

\[ = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2} \]

Corollary of page 35 from Rosenlicht tells that

\[ \sqrt{(a_1+b_1)^2 + (x_1+b_1)^2 + \ldots + (a_n+b_n)^2} \leq \sqrt{a_1^2+x_1^2} + \sqrt{b_1^2+b_2^2} + \ldots + \sqrt{b_n^2} \]

This applies to

\[ \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2} \]

Therefore, \[ |z_1 + z_2| \leq |z_1| + |z_2| \]

c) Suppose \[ z_1 = x_1 + iy_1 \] and \[ z_2 = x_2 + iy_2 \].

Then, \[ |z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = |x_1 x_2 + i(x_1 y_2 + y_1 x_2) + i^2 y_1 y_2| \]

\[ = |z_1 z_2| = |x_1 x_2 + y_1 y_2 + i(x_1 y_2 + y_1 x_2)| \]

\[ = \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2} \]

It is known that \[ |z_1| = \sqrt{x_1^2+y_1^2} \] and \[ |z_2| = \sqrt{x_2^2+y_2^2} \]

\[ |z_1 z_2| = \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2} = \sqrt{x_1^2 x_2^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + y_1^2 x_2^2 + 2x_1 y_2 x_1 y_1} \]

\[ = \sqrt{x_1^2 x_2^2 + x_2^2 y_1^2 + y_1^2 y_2^2} = \sqrt{x_1^2 (x_2^2 + y_2^2) + y_1^2 (x_2^2 + y_2^2)} = \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2)} \]

\[ = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2} = |z_1| |z_2| \]
29) $S$ complete subspace of $E$.

Since $S$ is complete, all Cauchy sequences must converge in $S$. So any convergent sequence is Cauchy and therefore must converge in $S$. So $S$ must be closed.

26) $S = E + tm : n, m \in \mathbb{Z}^+^3$. Any cluster point in $S$ has at least one sequence that converges to it. Any convergent sequence in $S$ is of the form

$$a_n = \frac{1}{n} + \frac{1}{m}, \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} + \frac{1}{m} = \frac{1}{m}.$$ 

So, the set of cluster points is

$$\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m} : m \in \mathbb{Z}^+ \}.$$

Taking the limit as $n$ goes to infinity yields 0 as a cluster point.

So our set of cluster points is

$$\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0 \}.$$
28 S is a closed subset of E, E metric space.
Suppose S contains all of its cluster points. Take any convergent series in S, \( p_n \rightarrow p \). The point limit of the convergent series \( p \) is itself a cluster point. Since any \( B(p, \epsilon) \) \( \epsilon > 0 \) must contain infinitely many points, since \( p_n \) is convergent. So every convergent sequence in S converges to a point in \( S \), hence \( S \) is closed.
Suppose \( S \) is closed. Let \( p \) be a cluster point of \( S \), by way of contradiction.
Suppose \( p \notin S \). Since \( p \) is a cluster point, any open ball around \( p \) contains infinitely many points. We can construct a sequence \( p_n \in B(p, \frac{1}{n}) \).
So for any \( \epsilon > 0 \) pick \( N \) such that \( \frac{1}{N} < \epsilon \). For \( n > N \), \( d(p_n, p) < \frac{1}{N} < \epsilon \).
So \( \lim p_n = p \) must converge. Since \( p \in S \), S must be open, a contradiction.
3b) a) Infinite subset of IR with no cluster point: \( \mathbb{Z} \)

The set of integers is an infinite subset of IR. To have a cluster point, we need to find a point \( p \) s.t. an open ball centered at \( p \) contains infinitely many points. However, for any finite radius ball, we have finitely many integers contained in the ball.

Therefore, the set of integers has no cluster points.

b) A complete metric space, that is bounded but not compact: \( E=(0,1) \)

The metric space is bounded but not compact. If we take the open cover \( E \cup \bigcup_{x \in (0,1)} B(x,r), r < 1 \), there is no finite subcover.

Only Cauchy sequences in the metric space are in the form \( \frac{1}{n}, 0, \ldots \) and obviously they converge to 0 \( \in E \). Therefore the metric space is complete.

c) A metric space none of its closed balls are complete: \( \mathbb{Q} \), \( d(x,y) = |x-y| \)

Take any closed ball in the metric space: \( \overline{B(x,r)}, r^2 \leq \frac{|x-y|}{n} \) enough \( n \) for each \( r \).

\( y = x + \frac{\sqrt{2}}{n} \) is irrational for any \( n \), and in the closed ball for large enough \( n \) for each \( r \).

Therefore, we can take a Cauchy sequence in a closed ball of the metric space which does not converge in the metric space.
32) Union of finite # of subsets of a metric space is compact.

E is a metric space

Take a collection of compact subsets of E: So CE, Si compact in E.

Let \(S = \bigcup_{i=1}^{n} S_i\), and let \(\bigcup_{j=1}^{m} U_j\) be an open cover of S.

For each \(i = 1, \ldots, n\)

Since we have \(S_i \subset S\) and \(S_i\) is compact, there must be a finite subcover

\[S_i \subset \bigcup_{j \in J_i} U_j\]

Now consider the union \(\bigcup_{i=1}^{n} J_i\), since it is finite and \(J_i\) are finite sets, this set is finite. Call this union K.

Then \(\bigcup_{j \in K} U_j\) is a finite subcover containing \(S = \bigcup_{i=1}^{n} S_i\).

Therefore S is compact.

33) E is a compact metric space, \(\forall i \in I\), a collection of open subsets of E s.t.

\(\bigcup_{i \in I} U_i = E\). We want to show \(\exists \varepsilon > 0\) s.t. any closed ball in E with radius \(\varepsilon\) is entirely contained in at least one \(U_i\).

Suppose the claim is not true: For each \(\varepsilon > 0\), there exists a closed ball with radius \(\varepsilon\) which is not contained in any of \(U_i\)’s.

Take \(\varepsilon = 1/n\), due to our assumption, for each n there is at least one closed ball \(\overline{B}(x_n, 1/n)\) which is not contained in any of \(U_i\)’s.

Note that sequence of centers of balls, \(x_n\), has a convergent subsequence in E, since E is compact. Call that limit \(x\), \(x \in E\).

Then we know \(x \in U_i\) for some i. (we’ll call that set just U)

Take an open ball around \(x\) in U: \(B(x, \varepsilon) \subset U\), \(\varepsilon > 0\)

Take \(\varepsilon = 1/n\), \(d(x_n, x) < \frac{\varepsilon}{2}\)

Then we get \(\overline{B}(x_n, 1/n) \subset \overline{B}(x, \varepsilon) \subset U\), contradiction.