Homework

1) a) We need to show that for any $\epsilon > 0$, there exists $\delta > 0$ so that $p \in F$ and $d(p, p_0) < \delta$, then $d'(f(p), f(p_0)) < \epsilon$

Let us check $p_0 = 0$

If $p < 0$, then $d(p, 0) = p$ and $d'(f(p), f(0)) = d'(0, 0) = 0$

If $p \geq 0$, then $d(p, 0) = p$ and $d'(f(p), f(0)) = d'(p, 0) = p$

Choose $\epsilon > 0$ and $\delta > 0$, for every $\epsilon > 0$, there exists $d(p, 0) < \delta$ and $d'(p, 0) < \epsilon$
Discuss the continuity of the function $f: \mathbb{R} \to \mathbb{R}$ if $f$ is given by:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is not rational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers with no common divisors other than } \pm 1, \text{ and } q > 0. \end{cases}$$

Claim: $f$ is not continuous.

Proof: Pick any $x_0 \in \mathbb{R}$, is $f$ continuous at $x_0$? If it was then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. From (LUB 5.) on page 26, there exists a rational number $\frac{r}{s} \in B(x_0; \delta)$ for any $\delta > 0$. (we may assume $r$, $s$ have no common factors.) Now pick any irrational number $k \in \mathbb{R}$ and choose a positive integer $N$ large enough so that $\frac{r}{s} + \frac{k}{N} \in B(x_0; \delta)$.

If $x_0$ is irrational then $|f(x) - f(x_0)| = |f(x)| < \varepsilon$. Then letting $x = \frac{r}{s}$ we see that $|f(x)| = \frac{1}{s} < \varepsilon$, so picking $\varepsilon \leq \frac{1}{s}$ leads to a contradiction.

If $x = \frac{p}{q}$ then $|f(x) - f(x_0)| = |f(x) - \frac{1}{q}| < \varepsilon$.

Now let $x = \frac{r}{s} + \frac{k}{N}$ so that $|f(x) - \frac{1}{q}| = \frac{1}{q} < \varepsilon$ and once again pick $\varepsilon \leq \frac{1}{q}$. Thus, $f$ is not continuous.
Let $E, E'$ be metric spaces, $f : E \to E'$ a continuous function. Show that if $S$ is a closed subset of $E'$, then $f^{-1}(S)$ is a closed subset of $E$. Derive from this the results that if $f$ is a continuous real-valued function on $E$ then the sets \( \{ p \in E : f(p) \leq 0 \} \), \( \{ p \in E : f(p) > 0 \} \), and \( \{ p \in E : f(p) = 0 \} \) are closed.

**Proof**: We wish to show that $C(f^{-1}(S))$ is an open subset of $E$. Since $SCE'$ is closed, $C(S)CE'$ is open and from the proposition on page 70 we know that $f^{-1}(C(S))CE$ is open. It follows that:

\[
C(f^{-1}(S)) = \{ p \in E : f(p) \notin S \} = \{ p \in E : f(p) \in C(S) \} = f^{-1}(C(S)).
\]

Then it follows immediately that $f^{-1}(S)$ is closed.

Now since \( \{ x \in \mathbb{R} : x \leq 0 \} \), \( \{ x \in \mathbb{R} : x > 0 \} \), \( \{ 0 \} \) are all closed subsets of $\mathbb{R}$, it follows that if $f : E \to \mathbb{R}$ then \( \{ p \in E : f(p) \leq 0 \} \), \( \{ p \in E : f(p) > 0 \} \), and \( \{ p \in E : f(p) = 0 \} \) are closed subsets of $E$.\]
Given \( f: U \to V \), strictly increasing and onto, where \( U, V \subseteq \mathbb{R} \) are open or closed intervals.

Claim: \( f \) is continuous on \( U \).

Proof: Note that \( U, V \) are bounded, \( f \) is one-one.

\( \Rightarrow \) If \( U, V \) are closed then we can see that \( U, V \) are nonempty compact subsets of \( \mathbb{R} \).

Assume \( f \) is not continuous. Then by corollary 2 on pg. 78, \( f \) does not attain a maximum at any \( u \in U \) or attain a minimum at any \( u \in U \).

\( \Rightarrow \) Since \( V \) is a nonempty compact set \( a = \text{lub} V, b = \text{glb} V \) but \( f \) is onto so \( \exists a_0, b_0 \in U \) such that \( f(a_0) = a \) and \( f(b_0) = b \). Then, \( f(b_0) \leq V, f(a_0) \geq V \) for all \( v \in V \). Once again, since \( f \) is onto this means:

\[
\begin{align*}
&f(b_0) \leq f(u) \quad \text{for all } u \in U, \\
&f(a_0) \geq f(u)
\end{align*}
\]

A contradiction. Thus, \( f \) is continuous.

\( \Rightarrow \) Now consider the case when \( U, V \) are open.

Once again, assume \( f \) is not continuous.

(continued \( \Rightarrow \))
Then by the proposition on page 70, there must exist some open subset \( B \subset V \) such that
\[
\mathcal{F}^{-1}(B) = \{ u \in U : \mathcal{F}(u) \in B \}
\]
is not an open subset of \( U \). Now pick \( u_0 = \mathcal{F}^{-1}(B) \).
Then \( \mathcal{F}(u_0) \in B \), but since \( B \) is open \( \exists \varepsilon > 0 \)
such that \( |\mathcal{F}(u) - \mathcal{F}(u_0)| < \varepsilon \) for all \( u \in U \) this holds. But now if we pick \( \delta > 0 \) such that \( |u - u_0| < \delta \) then \( |\mathcal{F}(u) - \mathcal{F}(u_0)| < \varepsilon \) where \( u \in \mathcal{F}^{-1}(B) \).
\[
\Rightarrow \text{But this implies that } \mathcal{F}^{-1}(B) \text{ is open, a contradiction.} \quad \text{\color{red}{Thus } } \mathcal{F} \text{ must be continuous.} \]
(a) Prove that \( \sqrt{x^2} \) is continuous on \( \{x \in \mathbb{R} : x \geq 0\} \).
Proof: Let \( E = \{x \in \mathbb{R} : x \geq 0\} \) and pick \( x_0 \in E \). If \( f \) is continuous on \( E \) then for every \( \varepsilon > 0 \) we must find \( \delta > 0 \) such that if \( x \in E \) and \( |x - x_0| < \delta \) then \( |f(x) - f(x_0)| = |\sqrt{x^2} - \sqrt{x_0^2}| < \varepsilon \).
Using some algebraic manipulation we see that
\[
|\sqrt{x^2} - \sqrt{x_0^2}| = \left| \frac{(\sqrt{x^2} - \sqrt{x_0^2})(\sqrt{x^2} + \sqrt{x_0^2})}{\sqrt{x^2} + \sqrt{x_0^2}} \right| = \left| \frac{x - x_0}{\sqrt{x^2} + \sqrt{x_0^2}} \right| \leq \frac{|x - x_0|}{\sqrt{x_0}}
\]
If we let \( |x - x_0| < \varepsilon \) then \( |\sqrt{x^2} - \sqrt{x_0^2}| < \varepsilon \) for all \( x \in \mathbb{R} \) such that \( |x - x_0| < \varepsilon \cdot \sqrt{x_0} = \delta \). Thus \( \sqrt{x^2} \) is continuous on \( E \).

(b) Evaluate \( \lim_{{x \to 1}} \frac{x - 1}{\sqrt{x^2} - 1} \). Let \( f : \{1\} \to \mathbb{R} \) where \( f(x) = \frac{x - 1}{\sqrt{x^2} - 1} \). Clearly, \( 1 \) is a cluster point of \( \mathbb{R} \).
Using some algebra:
\[
\frac{x - 1}{\sqrt{x^2} - 1} \cdot \frac{\sqrt{x^2} + 1}{\sqrt{x^2} + 1} = \frac{(x - 1)(\sqrt{x^2} + 1)}{(x - 1)(\sqrt{x^2} + 1)} = \frac{\sqrt{x^2} + 1}{x - 1}
\]
Thus, \( \lim_{{x \to 1}} \frac{x - 1}{\sqrt{x^2} - 1} = \lim_{{x \to 1}} (\sqrt{x^2} + 1) = \lim_{{x \to 1}} \sqrt{x^2} + \lim_{{x \to 1}} 1 = \sqrt{1} + 1 = 2 \)
since \( \sqrt{x^2}, 1 \) are continuous functions on \( \mathbb{C}(\{1\}) \subset \mathbb{R} \), so this follows from the corollary on page 76, and the first paragraph on page 74.
9(c) By exercise 8 we define \( \lim_{x \to +\infty} \frac{x}{2x^2 + 1} = \lim_{y \to 0} g(y) \)
where \( g: (0, 1) \to \mathbb{R} \) and \( g(y) = f\left(\frac{1}{y}\right) = \frac{1}{2y^2 + 1} = \frac{y}{2 + y^2} \).

Clearly, \( \lim_{y \to 0} \frac{y}{2 + y^2} = 0 \). We now prove this:
For every \( \varepsilon > 0 \) we must find \( \delta > 0 \) such that if \( y \in (0, 1) \) and \( |y| < \delta \) then \( \left| \frac{y}{2 + y^2} \right| < \varepsilon \). But,
\[
\Rightarrow \left| \frac{y}{2 + y^2} \right| = \frac{|y|}{|2 + y^2|} \leq \frac{|y|}{2 + y^2} \leq |y| < \delta.
\]
Therefore, if we let \( \varepsilon = \delta \) and \( |y| < \varepsilon \) then \( \left| \frac{y}{2 + y^2} \right| < \varepsilon \) as desired.

10(a) \( f: \mathbb{E}^2 \to \mathbb{R} \) where \( f(x, y) = \begin{cases} \frac{1}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \)
We only need to check the origin for continuity.
Consider the convergent sequence: \( \lim_{n \to \infty} (\frac{1}{n}, 0) = (0, 0) \)
If \( f \) is continuous at \( (0, 0) \) then we must have
\[
\lim_{n \to \infty} f\left(\frac{1}{n}, 0\right) = f(0, 0) = 0 \quad \text{but} \quad f\left(\frac{1}{n}, 0\right) = n^2
\]
so the limit doesn't exist.

10(b) \( f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \)
\[
\lim_{n \to \infty} (\frac{1}{n}, \frac{1}{n}) = (0, 0) \quad \text{but} \quad \lim_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2} \neq f(0, 0).
\]
\( f \) is not continuous at \( (0, 0) \).
(c) \( f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \)

Once again \( f(x, y) \) is continuous at all points \((x, y) \neq (0, 0)\) so we need to check for continuity at the origin: \( f \) is continuous at \((0, 0)\) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( d((x, y), (0, 0)) < \delta \) then we have

\[ \left| \frac{xy^2}{x^2 + y^2} \right| < \varepsilon. \]

Note that \((x - y)^2 = x^2 - 2xy + y^2 \geq 0\) which implies that \(x^2 + y^2 \geq 2xy\). Then we write:

\[ \Rightarrow \left| \frac{xy^2}{x^2 + y^2} \right| = \frac{|xy|^2}{x^2 - 2xy + y^2} \leq \frac{|xy|^2}{2xy} = \left| \frac{y}{2} \right| < \varepsilon \]

we have \(|y| < 2 \varepsilon\) and we may also suppose \(|x| < 2 \varepsilon\) so that \(x^2 + y^2 < 8 \varepsilon^2 \Leftrightarrow d((x, y), (0, 0)) = \sqrt{x^2 + y^2} < 2\sqrt{2}\varepsilon\). Therefore, \( f \) is continuous at \((x, y) = (0, 0)\).

11(i) If \( f : E \to \mathbb{R}, g : E \to \mathbb{R} \) are continuous at \( p_0 \in E \), show \((f + g)(p) = f(p) + g(p) \) is continuous at \( p_0 \in E \).

Since \( f, g \) are continuous at \( p_0 \in E \) we can find \( \delta_1, \delta_2 > 0 \) such that if \( p \in E \) and \( d(p, p_0) < \min\{\delta_1, \delta_2\} \) then \( |f(p) - f(p_0)| < \frac{\varepsilon}{2} \) and \( |g(p) - g(p_0)| < \frac{\varepsilon}{2} \).

Then:\n
\[ |(f + g)(p) - (f + g)(p_0)| = |f(p) - f(p_0) + g(p) - g(p_0)| \]
\[ \leq |f(p) - f(p_0)| + |g(p) - g(p_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

for all \( p \in E \) such that \( d(p, p_0) < \min\{\delta_1, \delta_2\} \).
We now show that if \( g(p) \) is continuous at \( p_0 \in E \) then \( h : E \to \mathbb{R} \), \( h(p) = -g(p) \) is continuous at \( p_0 \in E \).

⇒ We know for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( p \in E \) and \( d(p, p_0) < \delta \) then \( |g(p) - g(p_0)| < \varepsilon \).

but, \( |g(p) - g(p_0)| = |(-1)(g(p) - g(p_0))| = |(-g(p)) - (-g(p_0))| = |h(p) - h(p_0)| < \varepsilon \). Thus, \( h(p) \) is continuous at \( p_0 \in E \).

Now to prove \( (f - g)(p) = f(p) - g(p) \) is continuous at \( p_0 \in E \) simply apply part (a) to \( f(p) + (-g(p)) \).

Without loss of generality, we may assume that \( f(p_0) \neq 0 \), \( g(p_0) \neq 0 \) since the proof is straightforward if \( f(p_0) = g(p_0) = 0 \), where \( p_0 \in E \).

Now find \( \delta_1, \delta_2 > 0 \) so that if \( |p - p_0| < \delta \) then \( |f(p) - f(p_0)| < \varepsilon \) and \( |g(p) - g(p_0)| < \min \left\{ \varepsilon, \frac{\varepsilon}{2|f(p_0)|} \right\} \) where \( \varepsilon > 0 \) is arbitrary.

since, \( |g(p) - g(p_0)| < \varepsilon \) then \( |g(p)| < |g(p_0)| + \varepsilon \).

⇒ \( |(f - g)(p) - (f - g)(p_0)| = |f(p) - f(p_0)| \)

\[ = |(f(p) - f(p_0))g(p) + (g(p) - g(p_0))f(p)| \]

\[ \leq |f(p) - f(p_0)| |g(p)| + |g(p) - g(p_0)| |f(p)| \]

\[ \leq |f(p) - f(p_0)| (|g(p_0)| + \varepsilon) + |g(p) - g(p_0)| |f(p)| \]

continued.⇒
Then: \[ |(g^2)(p) - (g^2)(p_0)| < \frac{\varepsilon}{2(|g(p_0)| + \varepsilon)} (|g(p_0)| + \varepsilon) + \frac{\varepsilon}{2 |S(p)|} |f(p_0)| \]
\[ = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for } d(p, p_0) < \delta. \]

Since \( \varepsilon > 0 \) was arbitrary we find that \( f \cdot g \) is continuous at \( p_0 \in E \).

(2) To prove that \( \frac{f}{g} \) is continuous at \( p_0 \in E \), we first prove \( \frac{1}{g} \) is continuous at \( p_0 \in E \) provided that \( g(p_0) \neq 0 \).

Since \( g(p) \) is continuous at \( p_0 \in E \) we can pick \( \delta \) such that \( |g(p)| = |g(p_0) - (g(p_0) - g(p))| \geq |g(p_0)| - |g(p) - g(p_0)| \)
\[ > |g(p_0)| - \frac{|g(p)|}{2} = \frac{|g(p_0)|}{2}. \]
Then we have:
\[ \left| \frac{1}{g(p)} - \frac{1}{g(p_0)} \right| = \frac{|g(p) - g(p_0)|}{|g(p)| |g(p_0)|} < \frac{|g(p)|^2}{|g(p)| \cdot \left( \frac{|g(p)|}{2} \right)} = \varepsilon. \]

Since \( |g(p)| > \frac{|g(p_0)|}{2} \) \( \Rightarrow \) \( \frac{1}{|g(p)|} < \frac{1}{\frac{|g(p)|}{2}} \). Then \( \frac{1}{g} \) is continuous at \( p_0 \in E \). Now we apply part (c) to \( \frac{f}{g} = f \cdot \left( \frac{1}{g} \right) \) to show that \( \frac{f}{g} \) is continuous at \( p_0 \in E \).
Let $f : E \to \mathbb{R}$ be a continuous function on a compact metric space $E$.

Claim: $f$ is bounded and attains a maximum.

Proof: Assume $f$ is not bounded. Then for each $n = 1, 2, 3, \ldots$ we can find $|f(p_n)| > n$. Since $E$ is compact there exists a convergent subsequence $\{p_{n_k}\}$ of $\{p_n\}$ with

$$\lim_{k \to \infty} p_{n_k} = p_0, \quad p_0 \in E.$$ 

Now using the fact that $f$ is continuous, implies:

$$\lim_{k \to \infty} f(p_{n_k}) = f(p_0).$$

Thus we can find a positive integer $N$ such that:

$$1 > |f(p_{n_k}) - f(p_0)|$$

$$\geq |f(p_{n_k})| - |f(p_0)|$$

$$> n_k - |f(p_0)| \Rightarrow 1 + |f(p_0)| > n_k$$

for an infinite number of positive integers, a contradiction.

Therefore, $f$ is bounded and nonempty, so we can find a sequence in $f(E)$ with:

$$\lim_{n \to \infty} f(q_n) = \text{l.u.b.}\{f(p) : p \in E\}.$$ 

Once again, since $E$ is compact there exists a convergent subsequence $\{q_{n_k}\}$ of $\{q_n\}$ with:

$$\lim_{k \to \infty} q_{n_k} = q_0, \quad q_0 \in E.$$

(continued $\Rightarrow$)
but since \( \lim_{n \to \infty} f(q_n) = \lim_{k \to \infty} f(q_{nk}) \) we must have \( f(q_0) = \text{lub} \{ f(p) : p \in E \} \). Therefore, \( f(q_0) \geq f(p) \) for all \( p \in E \). Thus, \( f(q_0) \) is the maximum value.

(14) Let \( S \) be a nonempty compact subset of a metric space \( E \) and \( p_0 \in E \).

Claim: \( \min \{ d(p_0, p) : p \in S \} \) exists.

Consider the function \( f : E \to \mathbb{R} \) where \( f(p) = d(p, p_0) \). By example 2 on page 69, \( f \) is continuous on \( E \) but by example 7, page 70 \( f \) is continuous on the metric space \( S \). Since \( f \) is continuous function (real-valued) on the nonempty compact metric space \( S \) by corollary 2 on page 78 \( f \) attains a minimum at some point \( p \in S \) so \( \min \{ d(p_0, p) : p \in S \} \) exists.

(b) Let \( S \) be a nonempty closed subset of \( E^n \) and \( p_0 \in E^n \).

Claim: \( \min \{ d(p_0, p) : p \in S \} \) exists.

Note if \( p \in S \) then \( \min \{ d(p_0, p) : p \in S \} = 0 \) so we may assume that \( p_0 \in \mathcal{C}(S) \). (continued...)
Now pick $\varepsilon > 0$ such that the closed ball $\overline{B(p_0, \varepsilon)} \subset \mathbb{R}^n$ contains points in $S$, i.e. $\overline{B(p_0, \varepsilon)} \cap S \neq \emptyset$.

Once again consider the continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ given by $f(p) = d(p_0, p)$. (Example 2, pg. 69).

Then $f$ is continuous on $\overline{B(p_0, \varepsilon)} \cap S$ as well. (Example 7, pg. 70).

Note that $\overline{B(p_0, \varepsilon)} \cap S$ is a closed and bounded subset of $\mathbb{R}^n$, hence compact.

Then, by corollary 2 on pg. 78 $f$ attains a minimum at some point in $\overline{B(p_0, \varepsilon)} \cap S$, i.e. $\min \{d(p_0, p) : p \in S\}$ exists.

Let $E$ be a nonempty compact metric space.

Claim: $\max \{d(p, q) : p, q \in E\}$ exists.

Proof: Clearly $E$ is bounded since it is compact and $\{d(p, q) : p, q \in E\}$ is bounded and nonempty.

Then we can find a sequence of points $\{(p_n, q_n)\}_{n=1}^{\infty}$ of $E$ such that:

$$\lim_{n \to \infty} d(p_n, q_n) = \text{l.u.b.} \{d(p, q) : p, q \in E\}.$$ (continued $\Rightarrow$)
Since $E$ is compact there exists convergent subsequences of $\{p_n\}$, $\{q_n\}$ where $\{p_{n_k}\}$, $\{q_{n_k}\}$ converge to some $p_0$, $q_0 \in E$, respectively.

Hence: $d(p_0, q_0) = \lim_{k \to \infty} d(p_{n_k}, q_{n_k})$

$= \lim_{n \to \infty} d(p_n, q_n)$

$= \text{L.u.b.} \{d(p, q) : p, q \in E\}$.

Thus, $\max\{d(p, q) : p, q \in E\}$ exists, as desired. □