HW 8  
Solutions  7/10

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29) Given a uniformly continuous function, we can, given any $\varepsilon > 0$, find a $\delta > 0$ such that $\forall p, q \in E, \quad d(x, y) < \delta \Rightarrow d(f(p), f(q)) < \varepsilon$. Hence, if we divide the interval $[a, b]$ into intervals of length $\delta$, then by the continuity of the function, there is some interval for which $d(p, p') < \delta \Rightarrow d(f(p), f(p')) < \varepsilon$ where $p$ is a division point. Then choosing $\varepsilon/2$, we can find another point $p$ such that $d(p, p') < \varepsilon/2$. Continuing for $\forall n$, we can construct a sequence of points whose function values converge to $\gamma$. By the compactness of the real interval $[a, b]$, we can choose a convergent subsequence of points with function values converging to $\gamma$. Hence, for the limiting value $q \in [a, b]$ where $f(q) = \gamma$.

30) $f: \mathbb{R} \to \mathbb{R}, f(x) = \text{polynomial of odd degree, with presumably no zero coefficients. Any polynomial in } x, x_1, x_2, \ldots, x_n \text{ with coefficients in } \mathbb{R} \text{ is continuous. Since } \mathbb{R} \text{ is connected, } f(\mathbb{R}) \text{ is connected and we can apply the intermediate value theorem. Any polynomial of odd degree has at least one real root.} \quad \therefore \text{There exists } p, q \in \mathbb{R} \text{ such that } f(p) < 0 \text{ and } f(q) > 0. \text{ Thus, we can then choose any } p' \in \mathbb{R} \text{ such that } f(p') = 0 \text{ and obtain any } f(p') \in \mathbb{R}. \text{ Thus } f(\mathbb{R}) = \mathbb{R}.$

30) Consider a circle at points interior to the closed interval in $E^2$. Assume the function is continuous and $1:1$. Then as we increase the points of the circle either clockwise or counter-clockwise, the function must be monotonic increasing or decreasing (otherwise it would not be $1:1$), and hence we find, for any point on the circle, there are nearby points whose function values cannot be made arbitrarily close. This implies discontinuity, a contradiction. Hence the function must not be $1:1$. 
3b) Conjecturing that the limit is \( f(x) = 0 \), it needs to be shown that \( |0 - x^n(1-x^n)| < \varepsilon \) for \( n \) sufficiently large, independent of \( x \).

we need the following fact: \( |x^n(1-x^n)| \leq \frac{x^n}{n!} \) on \([0,1]\). This can be shown by induction. For \( n = 1 \), \( x - x^2 \leq |x| \) is a true statement. Then we can assume its truth of its statement for \( n \). Then \( x^{n+1}(1+x) = x^n(1-x^n) < x^n(1-x^n) = x^n |x^n| < \frac{x^n}{n!} \) for all \( n \geq 1 \) on \([0,1]\).

Now we can say \( |0 - x^n(1-x^n)| = |x^n(1-x^n)| \leq \frac{1}{n!} < \frac{1}{n!} < \frac{1}{n} \). This last quantity can be made smaller than \( \varepsilon \) by choosing \( n > \frac{1}{\varepsilon} \).

Then we conclude \( x^n(1-x^n) \) converges uniformly to \( f(x) = 0 \).

3c) Given a sequence of functions \( f_1, f_2, f_3 \) on \([0,1]\), determine uniform convergence:

\( f_n(x) = \frac{x}{1 + nx^2} \); \( \lim_{n \to \infty} f_n(x) = 0 \)

Choose \( \varepsilon > 0 \) find a \( N \) s.t. \( |f_n(x) - f(x)| = \left| \frac{x}{1 + nx^2} \right| \leq \left| \frac{1}{n} \right| < \varepsilon \)

\( \Rightarrow \) \( N = \frac{1}{\varepsilon} \); uniformly convergent:

\( f_n(x) = \frac{nx}{1 + nx^2} \); \( \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1 + nx^2} = \frac{1}{x} \)
Choose \( \varepsilon > 0 \), find \( N \): 
\[
|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| = \left| \frac{nx^2 - 1 - nx^2}{1+nx^2} \right| \leq \left| \frac{1}{1+n} \right| \cdot \frac{1}{n} < \varepsilon
\]

\( \Rightarrow N = \frac{1}{\varepsilon} \) : uniformly convergent.

\[f_n(x) = \frac{nx}{1+nx^2}; \quad \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x}{2x^2n} = \lim_{n \to \infty} \frac{1}{2nx} = 0\]

Choose \( \varepsilon > 0 \), find \( N \): 
\[
|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - 0 \right| = \left| \frac{nx}{1+nx^2} \right|
\]

\[
\leq \left| \frac{n}{1+n^2} \right| < \varepsilon \quad \Rightarrow \quad \frac{N}{1+N^2} = \varepsilon; \quad N = \frac{1}{\sqrt{1-4\varepsilon^2}}, \quad N \text{ has a very specific condition on } \varepsilon, \quad \frac{2\varepsilon}{\varepsilon}.
\]

Since we cannot choose any \( \varepsilon \), it is not uniformly convergent.

35) Since \( f \) is uniformly continuous for a given \( \varepsilon \), \( d'(f(p), f(q)) \leq \varepsilon \) can be achieved for all points \( p, q \in \mathbb{E} \) by choosing \( d'(p, q) \leq \delta \). Hence, for a given \( \varepsilon \), points \( x \) and \( x + \frac{1}{n} \) are less than the distance \( \delta \) apart when \( |x - x + \frac{1}{n}| = \frac{1}{n} \) \( < \delta \) or \( n > \frac{1}{\delta} \). Note that by the uniform cont. of the \( f_n \), \( \delta \) does not depend on the point. Then we can conclude that for \( n > \frac{1}{\delta} = N \), \( d'(f(x + \frac{1}{n}), f(x)) \leq \varepsilon \). Since the metric of \( d' \) is by definition the distance between the maxima of the functions, which will be less than \( \frac{1}{n} \) for all \( f_n(x), n > N \), here we supposed \( f(x + \frac{1}{n}) \) converging to \( f(x) \).

36) \( x, x/2, x/3, x/4, \ldots \) attain uniform convergence in \( \mathbb{R}^2 \).

\[
\lim_{n \to \infty} f_n(x) = \frac{x}{n} = 0.
\]

Choose an \( \varepsilon > 0 \), find an \( N \) s.t. \( |f_n(x) - f(x)| < \varepsilon, \forall n > N \).

\[
\Rightarrow \frac{1}{n} - 0 = \frac{1}{n} \leq \varepsilon \quad \Rightarrow \quad N = \frac{1}{\varepsilon}. \quad \text{As } N \text{ must depend on both } x \text{ and } \varepsilon, \text{ and since this shows } f_n \text{ converges at different rates depending on } x, \text{ it is not uniformly convergent.}
37) Given the two uniformly convergent sequences of functions $f_n$ and $g_n$, we know that for all $p \in E$, when $n > N_1$, $d'(f_n, f) < \varepsilon_1$ and $n > N_2$ ensures $d'(g_n, g) < \varepsilon_2$. Take $N = \max \{N_1, N_2\}$. The assertion is that this $N$ is sufficient to ensure

$$d'(f_n + g_n, f + g) = \max_{p \in E} d'(f_n + g_n, f + g) \leq \varepsilon_1 + \varepsilon_2 \leq \varepsilon$$

where $d'$ is the usual metric of the reals and it is understood that it is relative to our choice of $p$. Then $\max_{p \in E} d'(f_n + g_n, f + g) < \varepsilon_1 + \varepsilon_2 = \varepsilon$ for $n > N$.

The product $f_n g_n$ will also converge uniformly given $f_n, g_n$, too. For then $\max_{p \in E} d'(f_n g_n, f g) = \max_{p \in E} d'(f_n g_n - f g) = \max_{p \in E} d'(f_n - f)(g_n - g) = \max_{p \in E} d'(f_n - f) + d'(g_n - g) < \varepsilon_1 + \varepsilon_2 = \varepsilon$

$38)$ $E, E'$ metric spaces, $f_n : E \rightarrow E'$, $f_n$ bounded, $\sum_{n=1}^{\infty} f_n$ $\in E$. $E, E'$ is uniformly convergent.

To show: The limit function $f(x)$ is bounded.

Note: for bounded means that $f_n(E)$ is bounded; i.e., it can be contained in some ball. Set $\|f_n\| < M_n$, $M_n \in E$. Uniform convergence: for some $\varepsilon > 0$, there is an integer $N$ s.t. $d'(f_n(x), f(x)) < \varepsilon$, $n > N$, all $x \in E$. Proof: since $f_n(x)$ is bounded, then for any $x \in E$ and $n > N$ with uniform convergence we can state the following: $d'(M_n, f(x)) < \varepsilon$.

Although $f_n(x)$ may not attain this bound we then take $M = \max \{M_1, M_2, M_3, \ldots\}$.
In order to maintain convergence:
\[ d'(M, f(x)) < \varepsilon. \]
However, since $M$ is a constant, this places an upper bound on $f(x)$, the limit function. A similar argument can be made for a lower bound on $f(x)$, hence $f(x)$ is bounded as