Remark about problem 27

\( f(x) \) an odd degree polynomial.

Can assume that it look like

\[ f(x) = ax^n + \ldots, \quad a > 0. \quad \text{For } a < 0 \text{ the argument is similar.} \]

As \( x \to \infty \), \( f(x) \to \infty \) and as \( x \to -\infty \), \( f(x) \to -\infty \).

\( f(x) \) is continuous. Let \( p \in \mathbb{R} \) be any number. There exist \( x_1 \) so that \( f(x_1) > p \) and \( x_2 \) such that \( f(x_2) < p \). By the Intermediate Value Theorem there exists \( c \in \mathbb{R} \) with \( f(c) = p \).
Solution of problem 32

\[ f_0(x) = 0, \quad f_n(x) = \sqrt{x + f_{n-1}(x)}, \quad x \geq 0 \]

a) For every \( x \geq 0 \), \( f_n(x) \leq f_m(x) \).

\textit{Proof:} Assume it is true that \( f_{m-1}(x) \leq f_m(x) \). Then

\[ f_n(x) = \sqrt{x + f_{n-1}(x)} \leq \sqrt{x + f_m(x)} = f_{m+1}(x) \]

by the monotonicity of the root.

Now \( f_0(x) = 0 \leq f_1(x) \).

and by induction, the result follows.

b) The sequence \( f_n(x) \) is bounded for every \( x > 0 \).

Pretend for the moment that \( f_n(x) \) converges, for some \( x \). Then by the continuity of the square root and with \( f(x) = \lim_{n \to \infty} f_n(x) \),

\[ f(x) = \sqrt{x + f(x)} \text{ or} \]

\[ f^2(x) = x + f(x). \] 

This leads to
\[ f(x) = \frac{1}{2} \sqrt{\frac{1}{4} + x}, \quad \text{since} \]

only the positive root yields a non-negative function.

Now we claim that \( f_n(x) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + x} \)
for all \( x \geq 0 \) and all \( n = 1, 2, 3, \ldots \).

Clearly \( f_0(x) = 0 \leq \frac{1}{2} + \sqrt{\frac{1}{4} + x} \).

If \( f_n(x) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + x} \), then

\[ f_{n+1}(x) = \sqrt{x + f_n(x)} \leq \sqrt{x + \frac{1}{2} + \sqrt{\frac{1}{4} + x}} = \frac{1}{2} + \sqrt{\frac{1}{4} + x} \]

and by induction this claim is also proved to be true.

Thus for every \( x \geq 0 \), \( f_n(x) \) is a bounded monotone sequence which therefore converges to some function \( f(x) \) \( x \geq 0 \). We have seen before that

\[ f(x) = \sqrt{x + f(x)} \]
and hence \( f(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + x} \).