Final Exam for Calculus II, Math 1502, December 15, 2010

Name: 

Section: 

Name of TA: 

This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414…. Show your work, otherwise credit cannot be given. 

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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<thead>
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Problems related to Block 1:

I: (15 points) Compute with an error less than $10^{-3}$

$$\int_0^1 e^{x^4} \, dx .$$

II: a) (7 points) Compute the limit

$$\lim_{x \to 0} \frac{\log(1 + x) + (1 - x) - \cos x}{x^3}$$

b) (8 points) Does the improper integral

$$\int_0^1 \frac{1}{x^2} e^{-\frac{1}{x}} \, dx$$

exist? If yes, compute it.
Problems related to Block 2:

III: a) (7 points) Is the series
\[ \sum_{k=2}^{\infty} \frac{k \log k}{(\log k)^k} \]
convergent?

b) (8 points) Find the interval of convergence of the power series
\[ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} x^k \]

IV: (15 points) Solve the initial value problem
\[ y' + (x + 1)y = e^{-\frac{x^2}{2}} - x, \quad y(0) = 1. \]
Problems related to Block 3:

V: (20 points) Find a one-one parametrization for the solution of the systems below provided the solution exists:

\[
\begin{align*}
    x + 2y - 6z &= 2 \\
    2x + y + 3z &= 1 \\
    3x + y + 7z &= 1 .
\end{align*}
\]

\[
\begin{align*}
    x + 2y + 7z &= 1 \\
    x + 3y + 12z &= 1 \\
    -x + 4y + 23z &= 1 .
\end{align*}
\]

VI: (15 points) A plane in \( \mathbb{R}^3 \) has the parametric representation \( \vec{x}(s, t) = \vec{x}_0 + s\vec{v}_1 + t\vec{v}_2 \)

where \( \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), \( \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \). Find an equation for this plane.
Problems related to Block 4:

VII: (20 points) Consider the matrix

\[
A = \begin{bmatrix}
1 & 2 & 4 \\
2 & 3 & 5 \\
3 & 4 & 6 \\
\end{bmatrix}
\]

a) Find a basis for \(\text{Ker}(A)\) and a basis for \(\text{Img}(A)\),

b) Find a basis for \(\text{Ker}(A^T)\) and \(\text{Img}(A^T)\).

VIII: (15 points) Find the QR factorization of the matrix

\[
\begin{bmatrix}
1 & 6 & 4 \\
2 & 3 & -1 \\
2 & -6 & -10 \\
\end{bmatrix}
\]
IX: (15 points) Consider the curve given by the equation

\[ 8x^2 + 6xy = 1. \]

a) What is the type of the curve? Is it an ellipse or hyperbola?

b) Graph the curve in a qualitative fashion below. Indicate in the figure the directions of the eigenvectors.

X: (15 points) Find the eigenvalues and eigenvectors of the following matrices. What is their algebraic multiplicity, what is their geometric multiplicity?

\[ a) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b) \begin{bmatrix} 2 & 9 \\ 4 & 7 \end{bmatrix} \]
XI: (20 points) Using the superposition principle, solve the initial value problem given by the system

\[ \begin{align*}
    x' &= 2x + 9y \\
    y' &= 4x + 7y \\
    x(0) &= 1, \quad y(0) = 2
\end{align*} \]  

(0.1)

XII: (20 points) A sequence of numbers \( a_n, n = 0, 1, 2, \ldots \) satisfies the recursion relation

\[ a_{n+1} = 5a_n - 4a_{n-1}, \quad n = 0, 1, 2, \ldots \]

with the initial condition \( a_0 = 1, \ a_1 = 2 \). Find an expression for \( a_n \) for arbitrary \( n \).