Final Exam for Analysis I, Math 4317, December 13, 2010, Allowed time is 2 hours and 50 minutes. This is a closed book test but a cheat sheet is allowed. Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: Consider the real line \( \mathbb{R} \) with the metric \( d(x, y) := |x - y|, x, y \in \mathbb{R} \) and consider the subsets of \( \mathbb{R} \)
\[
A = (1, 2) , B = [1, 2) \cup \{3\} , C = [1, 2] .
\]
(4 points, no partial credit) Which ones are closed in \( \mathbb{R} \)?

(4 points, no partial credit) Which ones are open in \( \mathbb{R} \)?

(4 points, no partial credit) Which ones are compact?

2: Consider the metric space \( E = \{x \in \mathbb{R} : -1 < x < 1\} \) with the metric \( d(x, y) = |x - y|, x, y \in \mathbb{R} \) and the subsets
\[
A = (-1, 0) , B = (-1, 0] , C = [-1/2, 1/2] .
\]
(4 points, no partial credit) Which ones are closed in \( E \)?

(4 points, no partial credit) Which ones are open in \( E \)?

(4 points, no partial credit) Which ones are compact?

3: (16 points) Let \( p_1, p_2, p_3, \ldots \) be a sequence of points in a metric space \( E \) that converges to \( p \in E \). Prove that the set \( \{p, p_1, p_2, p_3, \ldots\} \) is closed.
4: (12 points) Prove or disprove by finding a counterexample the following statement: If $E$ is a metric space and $f : E \to E$ a continuous function then $f(S)$ is open for any open set $S \subset E$.

5: (12 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Show that the function

$$F(x) = \int_0^x f(t)dt$$

a uniformly continuous function on $\mathbb{R}$.

6: (16 points) Show, by directly using the definition of the logarithm, that

$$\lim_{x \to 0} x^\varepsilon \log(x) = 0$$

for every $\varepsilon > 0$. 
7: (20 points) Compute
\[
\lim_{n \to \infty} \frac{1 + 2^k + 3^k + \cdots + n^k}{n^{k+1}}.
\]
Here \(k \in \mathbb{R}\) is a positive number.

8: Let \(C\) be a compact subset of \(\mathbb{R}\). A function \(f : C \to \mathbb{R}\) is upper semicontinuous if and only if for all \(t \in \mathbb{R}\) the set \(\{x \in C : f(x) \geq t\}\) is closed.

a) (5 points) Prove that any continuous function \(f : C \to \mathbb{R}\) is upper semicontinuous.

b) (10 points) Prove that an upper semicontinuous function \(f : C \to \mathbb{R}\) is bounded.

c) Extra credit: (15 points) Prove that an upper semicontinuous function \(f : C \to \mathbb{R}\) attains its maximum value in the set \(C\).
Extra Credit: (30 points) Prove that the series

$$\sum_{n,m=1}^{\infty} \frac{1}{(n + m)!}$$

converges and determine its value.