This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414.... Show your work, otherwise credit cannot be given. **Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.**
I: (25 points) a) Consider the recursive sequence $a_{n+1} = \sqrt{2 + a_n}, n = 0, 1, 2 \ldots$ and $a_0 = 0$. Assuming that the sequence converges, compute its limit.

b) Compute the limit $\lim_{n \to \infty} a_n$ where

$$a_n = \frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}.$$

c) Express the number $0.\overline{123} = 0.123123 \ldots$ as a ratio of two integers.
(25 points) a) For what $a$ does the limit

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^a}$$

exist and is not zero?

Use any test to decide which of the following integrals exists:

$$a) \int_0^{\infty} \frac{1}{x + (x - 1)^2} \, dx , \quad b) \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{x(\ln x)^2} \, dx$$
III: (25 points) a) Solve the initial value problem

\[ y' + 3x^2y = x^2 \quad y(1) = 2 \]

b) (from Thomas) An aluminum beam was brought in from the outside cold into a machine shop where the temperature was held at 65°F. After 10 minutes, the beam warmed to 35°F and after another 10 minutes to 50°F. Use Newton’s law of cooling to compute the initial temperature of the beam.
IV: (25 points)

a) Consider the series
\[ \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a} \]
where \( a > 0 \). For which values of \( a \) is this series convergent and for which ones divergent.

b) Does the series
\[ \sum_{k=0}^{\infty} \sqrt{\frac{n+1}{n^3+2}} \]
converge?

c) Find \( n \) so that the partial sum \( s_n = \sum_{k=1}^{n} \frac{1}{k^4} \) estimates the value of the series \( \sum_{k=1}^{\infty} \frac{1}{k^4} \) with an error of at most \( 10^{-6} \).