This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414.... Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.
I: For the matrix

\[ A = \begin{bmatrix}
1 & 2 & 4 & -19 & 7 \\
2 & 5 & 5 & -26 & 9 \\
3 & 6 & 6 & -27 & 9 \\
\end{bmatrix} \]

a) (10 points) Find a basis for the column space of the matrix \( A \). Row reduction leads to

\[ \begin{bmatrix}
1 & 2 & 4 & -19 & 7 \\
0 & 1 & -3 & 12 & -5 \\
0 & 0 & 1 & -5 & 2 \\
\end{bmatrix} \]

The first, second and third column are pivotal columns and hence

\[ \begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}, \quad \begin{bmatrix}
2 \\
5 \\
6 \\
\end{bmatrix}, \quad \begin{bmatrix}
4 \\
5 \\
6 \\
\end{bmatrix} \]

form a basis for the column space.

b) (10 points) Find a basis for the null space of the matrix \( A \).

Calling the variables \( v, w, x, y, z \) we find that the variables \( y \) and \( z \) are free. Solving for the others yields

\[
\begin{bmatrix}
v \\
w \\
x \\
y \\
z \\
\end{bmatrix} = y \begin{bmatrix}
-7 \\
3 \\
5 \\
1 \\
0 \\
\end{bmatrix} + z \begin{bmatrix}
3 \\
-1 \\
-2 \\
0 \\
1 \\
\end{bmatrix}
\]

Hence,

\[ \begin{bmatrix}
-7 \\
3 \\
5 \\
1 \\
0 \\
\end{bmatrix}, \quad \begin{bmatrix}
3 \\
-1 \\
-2 \\
0 \\
1 \\
\end{bmatrix} \]

are a basis for the null space of \( A \). Note that the dimensions add up to the number of columns.
II: Consider the subspace $S$ of $\mathbb{R}^3$ spanned by the vectors
\[
\begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix},
\begin{bmatrix}
3 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
-2 \\
12
\end{bmatrix}
\]
a) (10 points) Find a basis for this subspace.
Again, row reduction of the matrix
\[
\begin{bmatrix}
1 & 3 & 1 \\
0 & 1 & -2 \\
2 & 1 & 12
\end{bmatrix}
\]
leads to
\[
\begin{bmatrix}
1 & 3 & 1 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]
and hence the first two columns are pivotal and
\[
\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}
\]
form a basis.
b) (10 points) Now, consider the subspace $T$ of $\mathbb{R}^3$ spanned by the vectors
\[
\begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix},
\begin{bmatrix}
0 \\
-1 \\
5
\end{bmatrix},
\begin{bmatrix}
5 \\
2 \\
0
\end{bmatrix}
\]
How are $T$ and $S$ related? Once more row reducing the matrix
\[
\begin{bmatrix}
2 & 0 & 5 \\
1 & -1 & 2 \\
-1 & 5 & 0
\end{bmatrix}
\]
leads to

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

and the vectors

\[
\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}
\]

form a basis of \( T \). Note that

\[
\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \vec{v} - \vec{u}
\]

and

\[
\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} = 3\vec{u} - \vec{v}.
\]

Hence \( T = S \). The set of vectors \( \{\vec{u}, \vec{v}\} \) and \( \{\vec{x}, \vec{y}\} \) are different bases but span the same space.
III: a) (10 points) Compute the inverse of the matrix

\[
A = \begin{bmatrix}
1 & 3 & -1 \\
2 & 2 & 2 \\
3 & 1 & 1
\end{bmatrix}.
\]

We have to row reduce

\[
\begin{bmatrix}
1 & 3 & -1 & | & 1 & 0 & 0 \\
2 & 2 & 2 & | & 0 & 1 & 0 \\
3 & 1 & 1 & | & 0 & 0 & 1
\end{bmatrix}
\]

which yields the inverse

\[
\frac{1}{4} \begin{bmatrix}
0 & -1 & 2 \\
1 & 1 & -1 \\
-1 & 2 & -1
\end{bmatrix}
\]

b) (10 points) Find the third column of the matrix $B^{-1}$ where

\[
B = \begin{bmatrix}
-1 & -7 & -3 \\
2 & 15 & 6 \\
1 & 3 & 2
\end{bmatrix}
\]

without computing the other columns.

We have to compute $B^{-1} \vec{e}_3$ which means that we have solve $B\vec{x} = \vec{e}_3$. Hence we have to row reduce

\[
\begin{bmatrix}
-1 & -7 & -3 & 0 \\
2 & 15 & 6 & 0 \\
1 & 3 & 2 & 1
\end{bmatrix}
\]
which yields

\[
\begin{bmatrix}
1 & 7 & 3 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

and back substitution yields

\[
\begin{bmatrix}
3 \\
0 \\
-1
\end{bmatrix}
\].
IV: Compute all the eigenvalues and the corresponding eigenvectors of the matrix

\[
\begin{bmatrix}
6 & -2 \\
6 & -1
\end{bmatrix}
\]

The characteristic polynomial is

\[
\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0
\]

and hence the eigenvalues are 2, 3. The eigenvectors for the eigenvalue 2 are any non-zero multiple of the vector

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

The eigenvector for the eigenvalue 3 is any non-zero multiple of the vector

\[
\begin{bmatrix}
2 \\
3
\end{bmatrix}
\].

b) (10 points) The matrix

\[
\begin{bmatrix}
2 & 2 & -1 \\
1 & 3 & -1 \\
-1 & -2 & 2
\end{bmatrix}
\]

has the 1 as an eigenvalue. Find the other eigenvalues and the corresponding eigenvectors.

We have to compute the determinant of the matrix

\[
\begin{bmatrix}
2 - \lambda & 2 & -1 \\
1 & 3 - \lambda & -1 \\
-1 & -2 & 2 - \lambda
\end{bmatrix}
\]
which yields the characteristic polynomial

\[-\lambda^3 + 7\lambda^2 - 11\lambda + 5.\]

We know that 1 is a root (check!) and hence we can divide

\[-\lambda^3 + 7\lambda^2 - 11\lambda + 5 : (\lambda - 1) = -\lambda^2 + 6\lambda - 5.\]

Solving

\[0 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)\]

and hence the eigenvalues are 1 and 5 where 1 is a double root. To find the eigenvectors for the eigenvalue 1 we have to row reduce the matrix

\[
\begin{bmatrix}
1 & 2 & -1 \\
1 & 2 & -1 \\
1 & 2 & -1 \\
\end{bmatrix}
\]

which yields

\[
\begin{bmatrix}
-2 \\
1 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
1 \\
\end{bmatrix}
\]

as a basis for the corresponding eigenspace. For the eigenvalue 5 we row reduce the matrix

\[
\begin{bmatrix}
-3 & 2 & -1 \\
1 & -2 & -1 \\
-1 & -2 & -3 \\
\end{bmatrix}
\]

which yields any nonzero multiple of the vector

\[
\begin{bmatrix}
-1 \\
-1 \\
1 \\
\end{bmatrix}
\]

as eigenvectors.
V: (5 points each) Prove or find a counter example. Let $A$ be an $m \times n$ matrix.

a) If the columns are linearly independent then the matrix is invertible.

This is false. Take the $2 \times 1$ matrix

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

which is clearly not invertible.

b) If the columns are linearly independent and span $\mathbb{R}^m$ then $n = m$.

This is true. Since there are $n$ linearly independent columns we must have $n \leq m$. Since the columns are vectors in $\mathbb{R}^m$ and span $\mathbb{R}^m$ they must be a basis and hence $n = m$.

c) If the dimension of $\text{Nul}(A)$ is $n - 1$ then $m = 1$

This is false. Take the $2 \times 2$ matrix

\[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}.
\]
The null space consists of any multiple of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and hence the dimension of the null space is 1 but $m = 2$. The number of columns is NOT necessarily the dimension of the column space. The dimension of the column space is 1 in this example.

d) If the dimension of $Col(A) = n$ then $A$ is invertible.

This is false. Take the $2 \times 1$ matrix

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The dimension of the column space is 1 but the matrix is not invertible.