**Problem 1:** Find the speed, the tangential acceleration and the normal acceleration for the motion
\[ \vec{r}(t) = (t, t^2, t^2). \]
Compute also the curvature of the corresponding curve as a function of \( t \).

The velocity, resp. acceleration is
\[ \vec{v}(t) = (1, 2t, 2t), \quad \vec{a} = (0, 2, 2), \]
and
\[ |\vec{v}| \]
is the speed. The tangential acceleration is
\[ a_T = \frac{d}{dt} |\vec{v}(t)| = \frac{d}{dt} \sqrt{1 + 8t^2} = \frac{8t}{\sqrt{1 + 8t^2}}. \]
The normal acceleration is
\[ a_N = \sqrt{|\vec{a}|^2 - a_T^2} = \sqrt{8 - \frac{64t^2}{1 + 8t^2}} = \frac{\sqrt{8}}{\sqrt{1 + 8t^2}}. \]
The curvature can be found using the formula
\[ \kappa(t) = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3} = \frac{\sqrt{8}}{(1 + 8t^2)^{3/2}}. \]

Here are some explanations: The tangential and normal acceleration are defined by
\[ \vec{a} = a_T \vec{T} + a_N \vec{N}, \]
where
\[ \vec{T} = \frac{\vec{v}}{|\vec{v}|}, \quad \vec{N} = \frac{dT}{ds} = \frac{dT}{dt} \frac{dt}{ds} = \frac{dT}{dt} \]
and \( s \) is the length parametrization. Now compute:
\[ \vec{a} = \frac{d}{dt} \vec{v} = \frac{d}{dt} \left( \frac{dT}{dt} \frac{ds}{dt} \right) = \frac{d^2s}{dt^2} \vec{T} + \frac{dT}{dt} \left( \frac{ds}{dt} \right)^2 \]
so that
\[ a_T = \frac{d^2s}{dt^2} = \frac{d|\vec{v}|}{dt}, \]
and
\[ a_N = \kappa \left( \frac{ds}{dt} \right)^2, \]
recalling that the curvature is given by
\[ \kappa = \frac{|dT|}{ds}. \]
From this it follows that
\[ \kappa = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3}. \]
The formula
\[ \vec{a} = \frac{d^2s}{dt^2} \vec{T} + \kappa \left( \frac{ds}{dt} \right)^2 \vec{N} \]
is useful and intuitive.

**Problem 2:** Find the moment of inertia with respect to the \( x \) axis of a thin shell of mass \( \delta \) that is in the first quadrant of the \( xy \) plane and bounded by the curve \( r^2 = \sin 2\theta \).

The moment of inertia with respect to the \( x \) axis is
\[ \delta \int_{\text{Region}} y^2 dxdy. \]
It is reasonable to work this integral in polar coordinates. Note that \( \sin 2\theta > 0 \) only if \( 0 \leq \theta \leq \pi/2 \) and \( \pi \leq \theta \leq 3\pi/2 \). Being in the first quadrant requires \( 0 \leq \theta \leq \pi/2 \). The moment of inertia with respect to the \( x \) axis is now the integral
\[ \delta \int_0^{\pi/2} \int_0^{\sqrt{\sin 2\theta}} (r \sin \theta)^2 r dr d\theta. \]
The distance of the point \((x, y)\) to the \( x \) axis is \( y^2 \). Integrating with respect to \( r \) yields
\[ \delta \int_0^{\pi/2} \int_0^{\sqrt{\sin 2\theta}} (r \sin \theta)^2 r dr d\theta = \delta \int_0^{\pi/2} (\sin \theta)^4 (\cos \theta)^2 d\theta. \]
\[ = \delta \int_0^{\pi/2} (\sin \theta)^4 d\theta - \delta \int_0^{\pi/2} (\sin \theta)^6 d\theta = \delta \frac{3\pi}{24} - \delta \frac{5\pi}{25} = \delta \frac{\pi}{25}. \]

**Problem 3:** Compute the center of mass of a thin shell that is formed by the cone \((z - 2)^2 = x^2 + y^2\), \(0 \leq z \leq 2\).

The following solves the wrong problem, namely for the solid cone.

The tip of the cone is at \( z = 2 \) and the base is a disk of radius 2. We use cylindrical coordinates. By symmetry \( x_{CM} = y_{CM} = 0 \). Now
\[ z_{CM} = \frac{\int_0^{2\pi} \int_0^2 \int_{2-r}^{2-z} zdz r dr d\theta}{\int_0^{2\pi} \int_0^2 \int_{2-r}^{2-z} d z r dr d\theta}. \]
The numerator is
\[ \frac{1}{2} \int_0^{2\pi} \int_0^2 (2 - r)^2 r dr d\theta = \pi \int_0^2 (2 - r)^2 r dr = \frac{4\pi}{3}, \]
and the denominator is
\[ \int_0^{2\pi} \int_0^2 (2 - r) r dr d\theta = 2\pi \int_0^2 (2 - r) r dr = \frac{8\pi}{3}, \]
so that
\[ z_{CM} = \frac{1}{2}. \]
Now the solution for the problem as posed:

Again, we have that $x_{CM} = y_{CM} = 0$, as before. The $z$ coordinate is

$$ z_{CM} = \frac{\int_{\text{Surface}} z d\sigma}{\int_{\text{Surface}} d\sigma} $$

the density $\delta$ cancels. We have to parametrize the cone, and we use conveniently cylindrical coordinates,

$$ \vec{r}(\theta, r) = (r \cos \theta, r \sin \theta, 2 - r) $$

noting that on the cone $z = 2 - r$. The tangent vectors are given by

$$ \vec{r}_r = (\cos \theta, \sin \theta, -1) $$

and

$$ \vec{r}_\theta = (-r \sin \theta, r \cos \theta, 0) . $$

The surface element is

$$ d\sigma = |\vec{r}_r \times \vec{r}_\theta| dr d\theta = |(r \cos \theta, r \sin \theta, r)| dr d\theta = \sqrt{2} r dr d\theta . $$

Now we integrate:

$$ \int_{\text{Surface}} d\sigma = \int_0^{2\pi} \int_0^2 \sqrt{2} r dr d\theta = 4\pi \sqrt{2} . $$

$$ \int_{\text{Surface}} zd\sigma = \int_0^{2\pi} \int_0^2 (2 - r) \sqrt{2} r dr d\theta = 2\pi \sqrt{2} \int_0^2 (2 - r) r dr = \frac{8\pi}{3} \sqrt{2} . $$

Hence,

$$ z_{CM} = \frac{2}{3} $$

Note that the Center of Mass is higher for the shell than for the solid, which is reasonable.

**Problem 4:** Compute the line integral of the vector field

$$ \vec{F} = (xyz + 1, x^2z, x^2y)e^{xyz} $$

along the curve given in parametrized form by

$$ \vec{r}(t) = (\cos t, \sin t, t) , \ 0 \leq t \leq \pi . $$

The line integral looks complicated and it is advisable to use Stokes’s theorem. Computing the curl of $\vec{F}$ yields $(0, 0, 0)$ and hence, by Stokes’s theorem the line integral depends only on the end points. The straight line that connects these two points is

$$ \vec{r}(t) = (1 - t)(1, 0, 0) + t(-1, 0, \pi) = (1 - 2t, 0, t\pi) , \ 0 \leq t \leq 1 . $$

We compute

$$ \vec{F} \cdot \vec{r}' = (1, (1 - 2t)^2t\pi, 0) \cdot (-2, 0, \pi) = -2 $$

and integrating this from 0 to 1 yields

$$ -2 . $$

With a little bit of guesswork one finds that

$$ \vec{F} = \nabla f , \ f = xe^{xyz} $$
and
\[ f(-1, 0, \pi) - f(1, 0, 0) = -1 - 1 = -2 \]

**Problem 5:** Use the divergence theorem to compute the outward flux of the vector field
\[ \vec{F} = (x^2, y^2, z^2) \]
through the cylindrical can that is bounded on the side by the cylinder \( x^2 + y^2 = 4 \), bounded above by \( z = 1 \) and below by \( z = 0 \).

Again, we invoke an integral theorem, but this time the divergence theorem. One computes easily
\[ \text{div} \vec{F} = 2(x + y + z) \]
and we have to integrate this over the cylinder. Using cylindrical coordinates
\[ 2 \int_0^{2\pi} \int_0^2 \int_0^1 [r(\cos \theta + \sin \theta) + z]rdrd\theta = 4\pi . \]

One can try to compute the flux directly. For the flux through the top one has to integrate \((x^2, y^2, 1) \cdot (0, 0, 1)\) over the disk of radius 2, which yields \(4\pi\). The bottom disk is particularly easy since the normal vector is \((0, 0, -1)\) and the vector field is \((x^2, y^2, 0)\) so that the dot product vanishes. Hence there is no contribution. It remains to compute the flux through the side. The parametrization of the cylinder is
\[ \vec{r}(\theta, z) = (2 \cos \theta, 2 \sin \theta, z) \]
so that
\[ \vec{r}_\theta = (-2 \sin \theta, 2 \cos \theta, 0) \; , \; \vec{r}_z = (0, 0, 1) \]
and
\[ \vec{r}_\theta \times \vec{r}_z = 2(\cos \theta, \sin \theta, 0) \]
which obviously points outward. Now
\[ \vec{F} \cdot \vec{n}d\sigma = ((2 \cos \theta)^2, (2 \sin \theta)^2, z^2) \cdot 2(\cos \theta, \sin \theta, 0)d\theta dz = 8((\cos \theta)^3 + (\sin \theta)^3)d\theta dz \]
and
\[ 8 \int_0^1 \int_0^{2\pi} ((\cos \theta)^3 + (\sin \theta)^3)d\theta dz = 0 . \]