Problem 1: The sphere \( x^2 + y^2 + z^2 = 14 \) and the plane \( x + y - z = 0 \) intersect in a circle. Find the line tangent to this circle at the point \((1, 2, 3)\).

The gradient of the function \( g(x, y, z) = x^2 + y^2 + z^2 \) is \( \nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle \), so that
\[
\nabla g(1, 2, 3) = \langle 2, 4, 6 \rangle.
\]
Likewise, the gradient of the function \( f(x, y, z) = x + y - z \) is \( \nabla f(x, y, z) = \langle 1, 1, -1 \rangle \) so that
\[
\nabla f(1, 2, 3) = \langle 1, 1, -1 \rangle.
\]
The plane tangent to the sphere at the point \((1, 2, 3)\) is perpendicular to \( \nabla g(1, 2, 3) \). The tangent line we are looking for is the intersection of these two planes. Hence the direction of this line is perpendicular to both \( \langle 2, 4, 6 \rangle \) and \( \langle 1, 1, -1 \rangle \). It is therefore natural to compute the cross product
\[
\langle 2, 4, 6 \rangle \times \langle 1, 1, -1 \rangle = \langle -10, 8, -2 \rangle.
\]
The lines must pass through the point \((1, 2, 3)\) and therefore the tangent line is given by all points of the form
\[
(1, 2, 3) + s\langle -10, 8, -2 \rangle.
\]

Problem 2: Find all the circles of radius 1 that are tangent to the curve \( \frac{x^2}{4} + y^2 = 2 \) at the point \((-2, 1)\).

Any circle of radius 1 is of the form \((x - a)^2 + (y - b)^2 = 1\). The goal is to find \(a, b\) so that the circles are tangent to the ellipse at the point \((-2, 1)\). To be tangent at the point \((-2, 1)\) means that the gradients of the function \( f(x, y) = \frac{x^2}{4} + y^2 \) and of the function \( g(x, y) = (x - a)^2 + (y - b)^2 \) evaluated at \((-2, 1)\) are parallel. Since
\[
\nabla f(x, y) = \langle \frac{x}{2}, 2y \rangle , \quad \nabla g(x, y) = \langle 2(x - a), 2(y - b) \rangle
\]
so that
\[
\nabla f(-2, 1) = \langle -1, 2 \rangle , \quad \nabla g(-2, 1) = \langle -4 - 2a, 2 - 2b \rangle.
\]
That the two vectors are parallel means that
\[
\langle -4 - 2a, 2 - 2b \rangle = \lambda \langle -1, 2 \rangle
\]
which when written out \(4 + 2a = \lambda \) and \(2 - 2b = 2\lambda \). This means that
\[
a = -2 + \frac{\lambda}{2} , \quad b = 1 - \lambda.
\]
We also know that \((-2 - a)^2 + (1 - b)^2 = 1\), since the point \((-2, 1)\) must be on the circle. So it must be that
\[
1 = (-2 - a)^2 + (1 - b)^2 = \frac{5\lambda^2}{4}.
\]
so that \( \lambda = \pm \frac{2}{\sqrt{5}} \). Thus, there are two circles which touches the ellipse, one on the ‘inside’ and one on the ‘outside’. They are

\[
(x + 2 - \frac{1}{\sqrt{5}})^2 + (y - 1 + \frac{2}{\sqrt{5}})^2 = 1
\]

and

\[
(x + 2 + \frac{1}{\sqrt{5}})^2 + (y - 1 - \frac{2}{\sqrt{5}})^2 = 1
\]

**Problem 3:** Find the distance between the curve \( x^2 - xy + y^2 = 1 \) and the line \( x + y = 10 \). What is the point closest to the line?

The line of distance between the curve \( x^2 - xy + y^2 = 1 \) and the line \( x + y = 10 \) must be perpendicular to both, the curve and the line. The vector normal to the line, which is the same as the gradient of the function \( x + y \), is \( \langle 1, 1 \rangle \). Thus we have to find a point \((u, v)\) on the curve \( x^2 - xy + y^2 = 1 \) where the gradient of the function \( f(x, y) = x^2 - xy + y^2 \) is parallel to \( \langle 1, 1 \rangle \). Now

\[
\nabla f(x, y) = \langle 2x - y, -x + 2y \rangle
\]

and we have to solve

\[
2x - y = \lambda, \quad -x + 2y = \lambda
\]

for \( x \) and \( y \). This is easily done and we get \( (x, y) = \lambda(1, 1) \). Now we have to choose \( \lambda \) so that the point \( \lambda(1, 1) \) is on the curve, i.e., \( \lambda^2 = 1 \) and hence \( \lambda = \pm 1 \). The curve is an ellipse and one point is closest to the line whereas the other is farthest from the line. Now we compute the distance between the line \( x + y = 1 \) and the point \((1, 1)\). The line passing through the point \((1, 1)\) and is normal to the line \( x + y = 10 \) is given by

\[
(1, 1) + s(1, 1)
\]

(Why?) and it intersects the line \( x + y = 10 \) at the point \((5, 5)\), i.e., \( s = 4 \). But the distance between the points \((1, 1)\) and \((5, 5)\) is \( 4\sqrt{2} \). Similarly the distance of the point \((-1, -1)\) to the line is \( 6\sqrt{2} \) which is larger. Hence the distance is \( 4\sqrt{2} \) and the point on the ellipse that is closest to the line is \((1, 1)\).