Potential Theory on Berkovich Spaces
Lecture 1: The Berkovich Projective Line

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Arizona Winter School on $p$-adic Geometry
March 2007
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The ultimate goal (which we will unfortunately not say much about) is to treat archimedean and non-archimedean analytic spaces in a unified way, and to make precise Arakelov’s analogy between intersection theory on arithmetic surfaces and potential theory on Riemann surfaces.
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Notation

- $K$: an algebraically closed field which is complete with respect to a nontrivial non-archimedean absolute value (e.g. $K = \mathbb{C}_p$)
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- **$\tilde{K}$**: the residue field of $K$
- **$B(a, r)$**: the closed disk $\{ z \in K : |z - a| \leq r \}$ of radius $r$ about $a$ in $K$. Here $r$ is any positive real number, and sometimes we allow the degenerate case $r = 0$ as well. If $r \in |K^*|$ we call the disk rational, and if $r \not\in |K^*|$ we call it irrational.
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- $B(a, r)^-$: the open disk $\{z \in K : |z - a| < r\}$ of radius $r$ about $a$ in $K$. 

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The Berkovich affine line $A^1_{Berk}$ over $K$ is a locally compact, Hausdorff, and uniquely path-connected topological space which contains $K$ as a dense subspace. The Berkovich projective line $P^1_{Berk}$ is obtained by adjoining a point $\infty$ to $A^1_{Berk}$. One can view $P^1_{Berk}$ as a profinite $\mathbb{R}$-tree. This allows one to define a Laplacian operator on $P^1_{Berk}$ which comes from the usual Laplacian on a finite graph. The tree structure also leads to a good theory of harmonic and subharmonic functions which closely parallels the classical theory over $\mathbb{C}$.
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A multiplicative seminorm on a ring $A$ is a function $|\cdot|_x : A \to \mathbb{R}_{\geq 0}$ satisfying:

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As a set, $\mathbb{A}^1_{\text{Berk}, K}$ consists of all multiplicative seminorms on the polynomial ring $K[T]$ which extend the usual absolute value on $K$. 
### Remarks

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Multiplicative seminorms (continued)

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3. We will usually omit explicit reference to the ground field $K$, writing $\mathbb{A}^1_{\text{Berk}}$. 
Topology on $\mathbb{A}^1_{\text{Berk}}$

Definition

The topology on $\mathbb{A}^1_{\text{Berk},K}$ is defined to be the weakest one for which $x \mapsto |f|_x$ is continuous for every $f \in K[T]$. 

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Topology on $\mathbb{A}^1_{\text{Berk}}$

**Definition**

The topology on $\mathbb{A}^1_{\text{Berk}}, K$ is defined to be the weakest one for which $x \mapsto |f|_x$ is continuous for every $f \in K[T]$.

Explicitly, a fundamental system of open neighborhoods is given by open sets of the form

$$\{ x \in \mathbb{A}^1_{\text{Berk}} : \alpha_i < |f_i|_x < \beta_i \}$$

with $f_1, \ldots, f_m \in K[T]$ and $\alpha_i, \beta_i \in \mathbb{R}$ ($i = 1, \ldots, m$).
The definition of $\mathbb{A}^1_{\text{Berk}}$ can be motivated by the following observations:

Every multiplicative seminorm on $\mathbb{C}[T]$ which extends the usual absolute value on $\mathbb{C}$ is of the form $f \mapsto |f(z)|$ for some $z \in \mathbb{C}$ (by the Gelfand-Mazur theorem), and the corresponding space $\mathbb{A}^1_{\text{Berk}}$, $\mathbb{C}$ is homeomorphic to $\mathbb{C}$.

When $K$ is non-archimedean, there are many more multiplicative seminorms on $K[T]$ than just the ones given by evaluation at a point of $K$.

Example
Fix a closed disk $B(a,r) = \{z \in K : |z-a| \leq r\}$ in $K$, and define $\| \|_{B(a,r)}$ by $\|f\|_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)|$.

Then $\| \|_{B(a,r)}$ is a multiplicative seminorm on $K[T]$ (by Gauss' lemma).
Motivation for the definition of $\mathbb{A}^1_{\text{Berk}}$

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Then $| \cdot |_{B(a,r)}$ is a multiplicative seminorm on $K[T]$ (by Gauss’ lemma).
The set of all (possibly degenerate) disks $B(a, r)$ therefore embeds naturally into $\mathbb{A}_Berk^1$. 

In particular, $K$ embeds into $\mathbb{A}_Berk^1$ as the set of disks of radius zero, and is dense in the Berkovich topology.

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If $a, a'$ are distinct points of $K$, one can visualize the unique path in $\mathbb{A}^1_{\text{Berk}}$ from $a$ to $a'$ as follows:
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- Start increasing the “radius” of the degenerate disk $B(a, 0)$ until we have a disk $B(a, r)$ which also contains $a'$.
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- This disk can also be written as $B(a', s)$ with $r = s = |a - a'|$. 

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\( \mathbb{A}^1_{\text{Berk}} \) is uniquely path-connected

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- Now decrease $s$ until the radius reaches zero and we have the degenerate disk $B(a', 0)$.
- In this way we have “connected up” the totally disconnected space $K$ by adding points corresponding to closed disks!
Nested sequences of closed disks

In order to obtain a **compact** space from this construction, it is usually necessary to add even more points.
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- For example, the field $\mathbb{C}_p$ is not spherically complete: this means that there are decreasing sequences of closed disks in $\mathbb{C}_p$ having empty intersection.
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- For example, the field $\mathbb{C}_p$ is not *spherically complete*: this means that there are decreasing sequences of closed disks in $\mathbb{C}_p$ having empty intersection.
- We need to add points corresponding to such sequences in order to obtain a compact space, since if $\{B(a_n, r_n)\}$ is any decreasing nested sequence of closed disks, the map

$$f \mapsto \lim_{n \to \infty} |f|_{B(a_n, r_n)}$$

defines a multiplicative seminorm on $\mathbb{K}[T]$ extending the usual absolute value on $\mathbb{K}$.

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defines a multiplicative seminorm on $K[T]$ extending the usual absolute value on $K$.

- Two such sequences of disks with empty intersection define the same seminorm if and only if the sequences are cofinal.
According to a result of Berkovich, we have now described all points of $\mathbb{A}^1_{\text{Berk}}$:

\begin{align*}
\text{Theorem (Berkovich's Classification Theorem)} \\
\text{Every point } x \in \mathbb{A}^1_{\text{Berk}} \text{ corresponds to a nested sequence } \\
B(a_1, r_1) \supseteq B(a_2, r_2) \supseteq B(a_3, r_3) \supseteq \cdots \\
\text{of closed disks, in the sense that } \\
|f|_x = \lim_{n \to \infty} |f|_{B(a_n, r_n)}. \\
\end{align*}

Two such nested sequences define the same point of $\mathbb{A}^1_{\text{Berk}}$ if and only if either:

1. each has a nonempty intersection, and their intersections are the same; or
2. both have empty intersection, and the sequences are cofinal.
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**Theorem (Berkovich’s Classification Theorem)**

*Every point $x \in \mathbb{A}^1_{\text{Berk}}$ corresponds to a nested sequence $B(a_1, r_1) \supseteq B(a_2, r_2) \supseteq B(a_3, r_3) \supseteq \cdots$ of closed disks, in the sense that*

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**Type I:** $B$ is a point of $K$.

**Type II:** $B$ is a closed disk with radius belonging to $|K^*|$.

**Type III:** $B$ is an irrational disk with radius not belonging to $|K^*|$.

**Type IV:** $B = \emptyset$. 

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We will denote by $\zeta_{a,r}$ the point of $\mathbb{A}^1_{\text{Berk}}$ of type II or III corresponding to the closed or irrational disk $B(a, r)$. 
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Following the terminology introduced by Chambert-Loir, the distinguished point $\zeta_{\text{Gauss}} = \zeta_{0,1}$ in $\mathbb{A}^1_{\text{Berk}}$ corresponding to the Gauss norm

$$\left| \sum_{i=0}^n a_i T^i \right|_{\text{Gauss}} = \max |a_i|$$

on $K[T]$ will be called the Gauss point.
A visual representation of $\mathbb{P}^1_{\text{Berk}}$
Alternate representation of $\mathbb{P}^1_{Berk}$
Note that:

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- Some of the branches extend all the way to the bottom (terminating in points of type I), while others are “cauterized off” earlier and terminate at points of type IV. In any case, every branch terminates either at a point of type I or type IV.
Tangent directions

Definition

Let $x \in \mathbb{P}_\text{Berk}^1$. The space $T_x$ of tangent directions at $x$ is the set of equivalence classes of paths $\ell_{x,y}$ emanating from $x$, where $y$ is any point of $\mathbb{P}_\text{Berk}^1$ not equal to $x$. Two paths $\ell_{x,y_1}, \ell_{x,y_2}$ are equivalent if they share a common initial segment.
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There is a natural bijection between elements $\vec{v} \in T_x$ and connected components of $\mathbb{P}^1_{\text{Berk}} \setminus \{x\}$. 
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- There is a natural bijection between elements \( \vec{v} \in T_x \) and connected components of \( \mathbb{P}^1_{\text{Berk}} \setminus \{x\} \).
- We denote by \( U(x; \vec{v}) \) the connected component of \( \mathbb{P}^1_{\text{Berk}} \setminus \{x\} \) corresponding to \( \vec{v} \in T_x \).
Tangent directions

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- We denote by \( U(x; \vec{v}) \) the connected component of \( \mathbb{P}^1_{\text{Berk}} \setminus \{x\} \) corresponding to \( \vec{v} \in T_x \).
- The open sets \( U(x; \vec{v}) \) for \( x \in \mathbb{P}^1_{\text{Berk}} \) and \( \vec{v} \in T_x \) generate the topology on \( \mathbb{P}^1_{\text{Berk}} \).
For $a \in K$ and $r > 0$, write

$$\mathcal{B}(a, r)^{-} = \{ x \in \mathbb{A}_{\text{Berk}}^1 : |T - a|_x < r \} ,$$

$$\mathcal{B}(a, r) = \{ x \in \mathbb{A}_{\text{Berk}}^1 : |T - a|_x \leq r \} .$$
Berkovich disks

For \( a \in K \) and \( r > 0 \), write

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\mathcal{B}(a, r)^- = \left\{ x \in \mathbb{A}^1_{\text{Berk}} : |T - a|_x < r \right\}, \\
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\]

We call a set of the form \( \mathcal{B}(a, r)^- \) an open Berkovich disk in \( \mathbb{A}^1_{\text{Berk}} \), and a set of the form \( \mathcal{B}(a, r) \) a closed Berkovich disk in \( \mathbb{A}^1_{\text{Berk}} \).
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For $a \in K$ and $r > 0$, write

$$B(a, r)^- = \{ x \in \mathbb{A}^1_{Berk} : |T - a|_x < r \},$$
$$B(a, r) = \{ x \in \mathbb{A}^1_{Berk} : |T - a|_x \leq r \}.$$

- We call a set of the form $B(a, r)^-$ an open Berkovich disk in $\mathbb{A}^1_{Berk}$, and a set of the form $B(a, r)$ a closed Berkovich disk in $\mathbb{A}^1_{Berk}$.
- Similarly, we can define open and closed Berkovich disks in $\mathbb{P}^1_{Berk}$. 
For $a \in K$ and $r > 0$, write

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- We call a set of the form $\mathcal{B}(a, r)^{-}$ an \textit{open Berkovich disk} in $\mathbb{A}^1_{\text{Berk}}$, and a set of the form $\mathcal{B}(a, r)$ a \textit{closed Berkovich disk} in $\mathbb{A}^1_{\text{Berk}}$.

- Similarly, we can define open and closed Berkovich disks in $\mathbb{P}^1_{\text{Berk}}$.

- The intersection of a Berkovich open disk with $\mathbb{P}^1(K)$ is a (classical) open disk (and similarly for closed disks).
A Berkovich open disk
Lemma

Every open set $U(x; \vec{v})$ with $x$ of type II or III and $\vec{v} \in T_x$ is a Berkovich open disk, and conversely.
Simple domains

**Lemma**

*Every open set $U(x; \vec{v})$ with $x$ of type II or III and $\vec{v} \in T_x$ is a Berkovich open disk, and conversely.*

Finite intersections of Berkovich open disks in $\mathbb{P}^1_{\text{Berk}}$ are called simple domains, and they form a fundamental system of open neighborhoods for the topology on $\mathbb{P}^1_{\text{Berk}}$. 
A simple domain $V$ in $\mathbb{P}^1_{\text{Berk}}$ has a finite set $x_1, \ldots, x_n$ of boundary points, and a corresponding finite set $\vec{v}_1, \ldots, \vec{v}_n$ of ends, which are the inward-pointing tangent directions:
A simple domain $V$ in $\mathbb{P}^1_{Berk}$ has a finite set $x_1, \ldots, x_n$ of boundary points, and a corresponding finite set $\vec{v}_1, \ldots, \vec{v}_n$ of ends, which are the inward-pointing tangent directions:
The tangent directions $\vec{v} \in T_{\zeta_{\text{Gauss}}}$ correspond bijectively to elements of $\mathbb{P}^1(\tilde{K})$, the projective line over the residue field of $K$. 
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Equivalently, elements of \( T_{\zeta_{\text{Gauss}}} \) correspond to the open disks of radius 1 contained in the closed unit disk \( B(0,1) \), together with the open disk

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B(\infty,1) := \mathbb{P}^1(K) \setminus B(0,1).
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B(\infty, 1)^- := \mathbb{P}^1(K) \setminus B(0,1).
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The correspondence between elements of \( T_{\zeta_{\text{Gauss}}} \) and open disks is given explicitly by \( \vec{v} \mapsto U(\zeta_{\text{Gauss}}; \vec{v}) \).
More generally, for each point $x = \zeta_{a,r}$ of type II, the set $T_x$ of tangent directions at $x$ is (non-canonically) isomorphic to $\mathbb{P}^1(\tilde{K})$: there is one tangent direction going “up” to infinity, and the other tangent directions correspond to open disks $B(a', r)^-$ of radius $r$ contained in $B(a, r)$. 
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For $x \in \mathbb{P}^1_{\text{Berk}}$, we have:

$$\left| T_x \right| = \begin{cases} \left| \mathbb{P}^1(\tilde{K}) \right| & \text{if } x \text{ of type II} \\ 2 & \text{if } x \text{ of type III} \\ 1 & \text{if } x \text{ of type I or type IV} \end{cases}$$
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Following notation introduced by Juan Rivera-Letelier, we write $H_{Berk}$ for the subset of $\mathbb{P}^1_{Berk}$ consisting of all points of type II, III, or IV.

- We refer to $H_{Berk}$ as “Berkovich hyperbolic space”.
Following notation introduced by Juan Rivera-Letelier, we write $H_{\text{Berk}}$ for the subset of $\mathbb{P}^1_{\text{Berk}}$ consisting of all points of type II, III, or IV.

- We refer to $H_{\text{Berk}}$ as “Berkovich hyperbolic space”.
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- We refer to $H_{\text{Berk}}$ as “Berkovich hyperbolic space”.
- We write $H^Q_{\text{Berk}}$ for the set of type II points, and $H^R_{\text{Berk}}$ for the set of points of type II or III.
- The subset $H^Q_{\text{Berk}}$ is dense in $\mathbb{P}^1_{\text{Berk}}$. 
Define the **diameter function** \( \text{diam} : \mathbb{A}^1_{\text{Berk}} \to \mathbb{R}_{\geq 0} \) by setting \( \text{diam}(x) = \lim r_i \) if \( x \) corresponds to the nested sequence \( \{B(a_i, r_i)\} \).

- If \( x \in H^R_{\text{Berk}} \), then \( \text{diam}(x) \) is just the diameter (= radius) of the corresponding closed disk.
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- If $x \in \mathbb{H}^\mathbb{R}_{\text{Berk}}$, then $\text{diam}(x)$ is just the diameter (= radius) of the corresponding closed disk.
- In terms of multiplicative seminorms, we have

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- In terms of multiplicative seminorms, we have
  \[
  \text{diam}(x) = \inf_{a \in K} |T - a|_x.
  \]

Because $K$ is complete, if $x$ is of type IV, then necessarily $\text{diam}(x) > 0$. Thus $\text{diam}(x) = 0$ for $x \in \mathbb{A}^1_{\text{Berk}}$ of type I, and $\text{diam}(x) > 0$ for $x \in H_{\text{Berk}}$. 
The space $\mathbb{A}^1_{\text{Berk}}$ is endowed with a natural partial order, defined by saying that

$$x \leq y \iff |f|_x \leq |f|_y \forall f \in K[T].$$
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In terms of (possibly degenerate) disks, if $x, y \in \mathbb{A}^1_{\text{Berk}}$ are points of type I, II, or III, we have $x \leq y$ if and only if the disk corresponding to $x$ is contained in the disk corresponding to $y$. 
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For each pair of points $x, y \in \mathbb{A}^1_{\text{Berk}}$, there is a unique least upper bound $x \lor y$ in $\mathbb{A}^1_{\text{Berk}}$ with respect to this partial order.
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For each pair of points $x, y \in \mathbb{A}^1_{\text{Berk}}$, there is a unique least upper bound $x \lor y$ in $\mathbb{A}^1_{\text{Berk}}$ with respect to this partial order.

Concretely, if $x = \zeta_{a,r}$ and $y = \zeta_{b,s}$ are points of type I, II or III, then $x \lor y$ is the point of $\mathbb{A}^1_{\text{Berk}}$ corresponding to the smallest disk containing both $B(a, r)$ and $B(b, s)$. 

Matthew Baker
Lecture 1: The Berkovich Projective Line
If $x, y \in H_{\text{Berk}}$ with $x \leq y$, define the **path metric**

$$\rho(x, y) = \log_v \frac{\text{diam}(y)}{\text{diam}(x)},$$

where $\log_v$ denotes the logarithm to the base $q_v$, with $q_v > 1$ a suitable constant.
If \( x, y \in \mathcal{H}_{Berk} \) with \( x \leq y \), define the path metric

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where \( \log_v \) denotes the logarithm to the base \( q_v \), with \( q_v > 1 \) a suitable constant.

For example, if \( K = \mathbb{C}_p \) and \( |p|_p = 1/p \), we would set \( q_v = p \) in order to have \( \{ \log_v |x|_p : x \in \mathbb{C}_p^* \} = \mathbb{Q} \).
If \( x, y \in \mathcal{H}_{\text{Berk}} \) with \( x \leq y \), define the **path metric**

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More generally, for \( x, y \in \mathcal{H}_{\text{Berk}} \) arbitrary, we define

\[
\rho(x, y) = \rho(x, x \lor y) + \rho(y, x \lor y).
\]

This gives an **metric** on \( \mathcal{H}_{\text{Berk}} \).
Remarks on the path metric on $\mathbb{H}_{\text{Berk}}$

- For closed disks $B(a, r) \subseteq B(a, R)$, we have
  
  $$\rho(\zeta_{a, r}, \zeta_{a, R}) = \log_{v} R - \log_{v} r.$$
Remarks on the path metric on $H_{\text{Berk}}$

- For closed disks $B(a, r) \subseteq B(a, R)$, we have
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- The points of type I should be thought of as infinitely far away from the points of $H_{\text{Berk}}$. 
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- The topology on $H_{\text{Berk}}$ defined by the metric $\rho$ is not the subspace topology induced from the Berkovich topology on $\mathbb{P}^1_{\text{Berk}}$. However, the inclusion map $H_{\text{Berk}} \hookrightarrow \mathbb{P}^1_{\text{Berk}}$ is continuous with respect to these topologies.
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- The group $\text{PGL}(2, K)$ of Möbius transformations acts continuously on $\mathbb{P}_1^{\text{Berk}}$ in a natural way compatible with the usual action on $\mathbb{P}_1(K)$, and this action preserves $H_{\text{Berk}}$. One can show that $\text{PGL}(2, K)$ acts via isometries on $H_{\text{Berk}}$, i.e.,
  $$\rho(M(x), M(y)) = \rho(x, y)$$
  for all $x, y \in H_{\text{Berk}}$ and all $M \in \text{PGL}(2, K)$. (This shows that the metric $\rho$ is canonical).
The canonical distance on $\mathbb{A}^1_{\text{Berk}}$

- The diameter function can be used to **extend the usual distance function** $|x - y|$ on $K$ to $\mathbb{A}^1_{\text{Berk}}$ in a natural way by setting

  $$[x, y]_{\infty} = \text{diam}(x \vee y)$$

for $x, y \in \mathbb{A}^1_{\text{Berk}}$. 
The canonical distance on $\mathbb{A}^1_{\text{Berk}}$

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- We call this extension the canonical distance (or Hsia kernel) on $\mathbb{A}^1_{\text{Berk}}$ (relative to infinity).
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- The diameter function can be used to extend the usual distance function $|x - y|$ on $K$ to $\mathbb{A}^1_{\text{Berk}}$ in a natural way by setting
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- If $x, y \in K$, then $[x, y]_\infty = |x - y|$.
The canonical distance on $\mathbb{A}^1_{\text{Berk}}$

- The diameter function can be used to extend the usual distance function $|x - y|$ on $K$ to $\mathbb{A}^1_{\text{Berk}}$ in a natural way by setting
  
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- We call this extension the canonical distance (or Hsia kernel) on $\mathbb{A}^1_{\text{Berk}}$ (relative to infinity).

- If $x, y \in K$, then $[x, y]_\infty = |x - y|$.

More generally:

If $x = \zeta_{a,r}$ and $y = \zeta_{b,s}$ are points of Type I, II, or III, then

$$[x, y]_\infty = \sup_{x_0 \in B(a,r)} \sup_{y_0 \in B(b,s)} |x_0 - y_0|.$$
Properties of $[x, y]_\infty$

1. For $y$ fixed, $[x, y]_\infty$ is continuous in $x$. 
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2. As a function of two variables, $[x, y]_\infty$ is merely upper semicontinuous.
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3. For all $x, y, z \in \mathbb{A}^1_{\text{Berk}}$, we have the ultrametric inequality

$$[x, y]_{\infty} \leq \max([x, z]_{\infty}, [y, z]_{\infty}),$$

with equality if $[x, z]_{\infty} \neq [y, z]_{\infty}$.
Properties of $[x, y]_\infty$

1. For $y$ fixed, $[x, y]_\infty$ is continuous in $x$.
2. As a function of two variables, $[x, y]_\infty$ is merely upper semicontinuous.
3. For all $x, y, z \in \mathbb{A}_\text{Berk}^1$, we have the ultrametric inequality

$$[x, y]_\infty \leq \max([x, z]_\infty, [y, z]_\infty),$$

with equality if $[x, z]_\infty \neq [y, z]_\infty$.
4. $[x, y]_\infty$ satisfies all of the axioms for an ultrametric except we have $[x, x]_\infty > 0$ for $x \in \mathbb{H}_{\text{Berk}}$. 