1. (10 points) (5P30) Find the area of the bounded region in the first quadrant cut off by $3 = (\sqrt{x} + \sqrt{y})^2$.

Solution:

$$A = \int_0^3 (\sqrt{3} - \sqrt{x})^2 \, dx$$

$$= [3x - 2\sqrt{2} \left(\frac{2}{3}\right) x^{3/2} + x^2 / 2]_0^3$$

$$= 9 - 4\sqrt{6} + 9/2$$

$$= \frac{27 - 4\sqrt{6}}{2}.$$
2. (10 points) (6.5.30) A pump rated at 1000 ft-lbs per second is used to pump 250 cubic feet of water (weighing 62.4 lbs per cubic foot) from a reservoir level $z = 0$ into a cylindrical tank with bottom 50 feet above the reservoir and radius 5 ft. Calculate how long this should take.

**Solution:** Notice that the tank will be filled to a height $H = 250/(25\pi) = 10/\pi$. The work required to pump the slice of water which ends up at height $h$ in the tank is approximately

$$(50 + h)(62.4)(25\pi)\Delta h.$$ 

Thus, the total work required is

$$W = \int_0^{\pi/10} (50 + h)(62.4)(25\pi) \, dh = 25(62.4)\pi[5\pi + \pi^2/200].$$ 

This would be given in ft-lbs, and if we divide by the rating of the pump, we should obtain the total time in seconds:

$$\frac{62.4\pi^2(125 + \pi/8)}{1000} = \frac{15.6\pi^2(500 + \pi/2)}{1000} = \frac{7.8\pi^2(1000 + \pi)}{1000} = 7.8\pi^2(1 + \pi/1000),$$

or about 77 seconds, or a little over a minute.
3. (10 points) (7.1.28) Compute the definite integral

\[ \int_{-2}^{0} 5^{-\theta} d\theta. \]

Solution: \( u = -\theta \ln 5. \)

\[
\int_{-2}^{0} e^{-\theta \ln 5} \, d\theta = -\frac{1}{\ln 5} \int_{\ln 25}^{0} e^{u} \, du = \frac{1}{\ln 5} e_{\ln 25}^{u} \bigg|_{0}^{24} = \frac{24}{\ln 5}.
\]
4. (10 points) (8.4.50) Solve the initial value problem.

\[ \begin{align*}
\sqrt{x^2 - 9} y' &= 1 \quad \text{for } 3 < x \\
y(5) &= \ln 3.
\end{align*} \]

**Solution:** Integrating \( y' = 1/\sqrt{x^2 - 9} \), we obtain

\[
y(x) - y(5) = \int_{5}^{x} \frac{1}{\sqrt{t^2 - 9}} \, dt = \frac{1}{3} \int_{5}^{x} \frac{1}{\sqrt{(t/3)^2 - 1}} \, dt = \int_{5/3}^{x/3} \frac{1}{\sqrt{u^2 - 1}} \, du.
\]

At this point, we make a trigonometric substitution \( \sec \theta = u \) according to which \( \cot \theta = 1/\sqrt{u^2 - 1} \) and \( du = \sec \theta \tan \theta \, d\theta \).

\[
y(x) = \ln(3) + \int_{5/3}^{x/3} \frac{1}{\sqrt{u^2 - 1}} \, du = \ln(3) + \int_{\sec^{-1}(x/3)}^{\sec^{-1}(5/3)} \sec \theta \, d\theta
\]

\[
= \ln(3) + \ln(\sec \theta + \tan \theta) \bigg|_{\sec^{-1}(x/3)}^{\sec^{-1}(5/3)}
\]

\[
= \ln(3) + \ln\left(\frac{x}{3} + \sqrt{x^2 - 9/3}\right) - \ln(3)
\]

\[
= \ln\left(\frac{x + \sqrt{x^2 - 9}}{3}\right).
\]
5. (15 points) (9.2.3) Solve the first order linear equation

\[ xy' + 3y = \frac{\sin x}{x^2} \] for \( 0 < x \).

**Solution:** Putting the equation into standard form, we get

\[ y' + \frac{3}{x}y = \frac{\sin x}{x^3}. \]

Thus, the integrating factor should be

\[ \mu = e^{\int \frac{3}{x} \, dx} = e^{3\ln x} = x^3. \]

Multiplying the equation by the integrating factor, we get

\[ x^3 y' + 3x^2 y = (x^3 y)' = \sin x. \]

Integrating, from some \( x_0 > 0 \) to \( x \),

\[ x^3 y - x_0^3 y(x_0) = -\cos x + \cos x_0. \]

Therefore,

\[ y = \frac{1}{x^3} \left[ -\cos x + \cos x_0 + x_0^3 y(x_0) \right]. \]

In this expression \( x_0 \) can be any positive constant, and \( y(x_0) \) can be taken to be any constant. Alternatively, we can write simply

\[ y = \frac{1}{x^3} \left[ -\cos x + C \right] \]

where \( C \) is an arbitrary constant.
6. (15 points) (10P18) Discuss the convergence of the sequence $(-4)^n/n!$.

**Solution:** The limit is zero (and it helps to know this beforehand—see Theorem 5 in Chapter 10 Section 1). As long as $n > 5$, we have

$$
\left| \frac{(-4)^n}{n!} \right| = \frac{4^n}{n!} < \frac{4^n}{5^n \cdot 5!} = \frac{5^5}{5!} \left( \frac{4}{5} \right)^n \to 0.
$$
7. (15 points) (10.7.27) Discuss the convergence of the series

\[ \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(x+2)^j}{j^{2j}}. \]

**Solution:** Denoting the \( j \)-th term by \( a_j \), we attempt to use the ratio test:

\[
\lim_{j \to \infty} \frac{|a_{j+1}|}{|a_j|} = \lim_{j \to \infty} \frac{|(x+2)(j+1)^{2j+1}|}{j^{2j}2} = \frac{|x+2|}{2}.
\]

Thus, the series converges absolutely for \(|x+2| < 2\) and diverges for \(|x+2| > 2\). When \( x = -4 \), the series becomes

\[ \sum \frac{-1}{j}. \]

This is the negative harmonic series and diverges, for example, by the integral test. When \( x = 0 \), the series becomes

\[ \sum \frac{(-1)^j}{j}. \]

This is the (negative) alternating harmonic series and converges by the alternating series test.
8. (15 points) (10.9.5) Find the Taylor series expansion for $\cos(5x^2)$ at $x = 0$ and discuss the convergence of the series.

**Solution:** We know

$$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}. $$

Therefore,

$$\cos(5x^2) = \sum_{j=0}^{\infty} \frac{(-25)^j}{(2j)!} x^{4j},$$

and this series converges (absolutely) for all $x$ since the series for $\cos x$ converges for all $x$. 