1. (20 points) Prove or disprove: The intersection of open sets is open. (Fully justify your answer.)

Solution: Disproof: Each of the intervals \((0, 1 + 1/j)\) for \(j = 1, 2, \ldots\) is open, but

\[ \bigcap_{j=1}^{\infty} (0, 1 + 1/j) = (0, 1] \]

is not open because there is no open ball \(B_r(1)\) with positive radius \(r\) and center \(x = 1\) which lies entirely in the intersection.
2. (20 points) (11A) Define the term *compact*.

Prove directly from the definitions (without using the Heine-Borel Theorem) that a compact set is closed.

**Solution:** A set $K$ is compact if any open cover of $K$ has a finite subcover.

We need to show the complement of $K$ is open. Let $K$ be a compact set and let $x$ be in the complement of $K$. Then notice that $\{\mathbb{R}^n \setminus B_r(x) \}_{r>0}$ is an open cover of $K$. Since $K$ is compact, this cover has a finite subcover:

$\{\mathbb{R}^n \setminus B_{r_1}(x), \mathbb{R}^n \setminus B_{r_2}(x), \ldots, \mathbb{R}^n \setminus B_{r_k}(x)\}.$

Taking $r = \min\{r_1, \ldots, r_k\} > 0$, we find that $K \subset \mathbb{R}^n \setminus B_r(x)$. Thus, $B_r(x) \subset K^c$, and it follows that the complement $K^c$ is open. Thus, $K$ is closed.
3. (20 points) Define the term *uniformly continuous*.

Prove the uniform limit of uniformly continuous functions is uniformly continuous.

**Solution:** Given a function $f : A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$, we say $f$ is *uniformly continuous* if given $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(x_2) - f(x_1)| < \epsilon \quad \text{whenever} \quad x_1, x_2 \in A \text{ with } |x_2 - x_1| < \delta.$$  

Proof: Let $f$ be the uniform limit of uniformly continuous functions $f_j$. That is, given $\epsilon > 0$, there is some $N$ such that

$$j \geq N \implies \text{dist}(f_j, f) < \epsilon.$$  

Let $\epsilon_0 > 0$. From the convergence above, there is some $N$ such that $\text{dist}(f_N, f) < \epsilon_0/3$. Since $f_N$ is uniformly continuous, there is some $\delta > 0$ such that $|f_N(x) - f_N(x_0)| < \epsilon_0/3$ whenever $x$ and $x_0$ are in $A$ with $|x - x_0| < \delta$.

Thus, if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \epsilon_0.$$  

This shows that $f$ is uniformly continuous.
4. (20 points) Write down the correct power series for the function \( \sin(x) \) with center of expansion at \( x = 0 \).

Determine the radius of convergence of this series, and clearly justify your answer.

**Solution:**

\[
\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.
\]

The radius of convergence is \( R = +\infty \). To see this, first note that

\[
\sum_{k=0}^{\infty} \left| (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right|
\]

is dominated by

\[
\sum_{j=0}^{\infty} \frac{|x|^j}{j!}.
\]

Applying the ratio test to this series, we find

\[
\frac{|x|^{j+1}}{(j+1)! \cdot |x|^j} = \frac{|x|}{j+1} \to 0.
\]

Thus, the series for sine is absolutely convergent for all \( x \), and the radius of convergence is +\( \infty \).
5. (20 points) (Theorem 27.9) Give a correct statement of the mean value theorem.

Use the mean value theorem to prove: If \( f, g \in C^0[a, b] \) and the derivatives \( f', g' : (a, b) \to \mathbb{R} \) exist with \( f' \equiv g' \), then there is some constant \( c \in \mathbb{R} \) such that \( f(x) = g(x) + c \) for \( x \in [a, b] \).

**Solution:** MVT: If \( \phi \in C^0[a, b] \) and the derivative \( \phi' : (a, b) \to \mathbb{R} \) exists, then there is some \( \xi \in (a, b) \) such that

\[
\phi'(\xi) = \frac{\phi(b) - \phi(a)}{b - a}.
\]

Consider the function \( \phi = f - g \). This function satisfies the hypotheses of the mean value theorem on every interval \([\alpha, \beta] \subset [a, b]\), and \( \phi' \equiv 0 \). Consequently, for some \( \xi \in (a, b) \),

\[
\frac{f(\beta) - g(\beta) - [f(\alpha) - g(\alpha)]}{\beta - \alpha} = \phi'(\xi) = 0.
\]

Therefore, \( \phi(x) = f(x) - g(x) \equiv c \) is constant. This means \( f(x) = g(x) + c \).