1. (25 points) (4-2.4) Let \( U = \{(u, v) : u, v > 0\} \) and show that \( X : U \to \mathbb{R}^3 \) given by
\[
X(u, v) = (u^2 - v^2, 0, 2uv)
\]
is a conformal coordinate on the surface \( S = X(U) \).

**Solution:** If \( p = X(u, v) \) and \((a, b), (c, d) \in T_{X^{-1}(p)}U\), then
\[
\langle (a, b), (c, d) \rangle_{T_{X^{-1}(p)}U} = \langle (a, b), (c, d) \rangle_{T_{X^{-1}(p)}\mathbb{R}^2} = ac + bd.
\]
On the other hand,
\[
X_u = (2u, 0, 2v) \quad \text{and} \quad X_v = (-2v, 0, 2u).
\]
Thus,
\[
E = 4(u^2 + v^2), \quad F = 0, \quad G = 4(u^2 + v^2),
\]
and
\[
\langle dX(a, b), dX(c, d) \rangle_{T_pS} = \langle aX_u + bX_v, cX_u + dX_v \rangle_{T_pS}
\]
\[
= acE + (ad + bc)F + bdG
\]
\[
= 4(u^2 + v^2)(ac + bd).
\]
Thus,
\[
\langle dX(a, b), dX(c, d) \rangle_{T_pS} = 4(u^2 + v^2)\langle (a, b), (c, d) \rangle_{T_{X^{-1}(p)}U}.
\]
This implies that \( X \) is conformal. Alternatively, the definition of isothermal parameters requires only that \( E = G \) and \( F = 0 \), which we have also shown above.

2. (25 points) (4-2.1,12) Let \( U = \{(u, v) \in \mathbb{R}^2 : |v| < u\} \). Find an isometry \( X : U \to S \) of \( U \) onto the conical surface
\[
S = \{(x, y, \sqrt{x^2 + y^2}) \in \mathbb{R}^3 : x > \cot(\pi \sqrt{2}/4)|y|\}.
\]

**Solution:**
\[
X(u, v) = \sqrt{\frac{u^2 + v^2}{2}} \left( \cos \left( \sqrt{2} \tan^{-1} \left( \frac{u}{v} \right) \right), \sin \left( \sqrt{2} \tan^{-1} \left( \frac{u}{v} \right) \right), 1 \right).
\]
The construction of this map was explained in class. In particular, the polar angle \( \theta \) is computed by matching circular arclengths:
\[
\sqrt{u^2 + v^2} \tan^{-1} \left( \frac{u}{v} \right) = \sqrt{\frac{u^2 + v^2}{2}} \theta.
\]
The length on the left is that of a circular arc in $\mathcal{U}$ with polar radius $\sqrt{u^2 + v^2}$, and the length on the image arc on the conical surface.

This map is easily seen to be one-to-one, onto, continuous, and has a continuous inverse. In fact, it is clear that the map is smooth as well as its inverse. Thus, $X$ is a diffeomorphism.

Computing the first fundamental form, we find

$$X_u = \frac{u}{\sqrt{2(u^2 + v^2)}} \left( \cos \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), \sin \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), 1 \right)$$

$$- \frac{v}{\sqrt{u^2 + v^2}} \left( - \sin \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), \cos \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), 0 \right).$$

$$X_v = \frac{v}{\sqrt{2(u^2 + v^2)}} \left( \cos \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), \sin \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), 1 \right)$$

$$+ \frac{u}{\sqrt{u^2 + v^2}} \left( - \sin \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), \cos \left( \sqrt{2} \tan^{-1} \left( \frac{v}{u} \right) \right), 0 \right).$$

Thus,

$$E = 1 = G, \quad F = 0,$$

and $X$ is an isometry.

3. (25 points) (4-3.3) Compute the Gauss curvature of the surface parameterized on $\mathbb{R}^2$ by

$$X(u, v) = (u, v, u^2 + v^2).$$

**Solution:**

$$X_u = (1, 0, 2u), \quad X_v = (0, 1, 2v),$$

so that

$$E = 1 + 4u^2, \quad F = 4uv, \quad G = 1 + 4v^2,$$

and

$$N = (-2v, -2u, 1)/\sqrt{1 + 4u^2 + 4v^2}.$$

It follows that

$$e = 2/\sqrt{1 + 4u^2 + 4v^2}, \quad f = 0, \quad g = 2/\sqrt{1 + 4u^2 + 4v^2},$$

and

$$K = \frac{eg - f^2}{EG - F^2} = \frac{4}{(1 + 4u^2 + 4v^2)^{3/2}}.$$
4. (25 points) (4-3.4) Explain why it is not possible to find a local isometry \( \phi : S \to \tilde{S} \) if

\[
S = \{(x, y, x^2 + y^2) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\},
\]

and

\[
\tilde{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 = 1\}.
\]

**Solution:** The second surface is an ellipsoid, which has Gauss curvature positive and (because it is a closed surface and the Gauss curvature is continuous) bounded away from zero. The first surface is the one from the previous problem and has points of arbitrarily small positive Gauss curvature as \( u^2 + v^2 \to \infty \). Thus, it is impossible for these points of small Gauss curvature on the paraboloid to map to points with the same Gauss curvature on the ellipsoid.