1. (25 points) (1-5.14) Let \( \gamma : (0, 1) \rightarrow \mathbb{R}^3 \) be a regular parameterized curve. If \( |\gamma(t)| \leq 3 \) for \( 0 < t < 1 \), and \( |\gamma(1/2)| = 3 \), show that the curvature \( k = k(1/2) \) at \( \gamma(1/2) \) satisfies
\[
k \geq \frac{1}{3}.
\]

**Solution:** Reparameterize the curve by arclength, let the reparameterization be denoted by \( \alpha \), and assume that \( \alpha(0) = \gamma(1/2) \). Since \( |\alpha(s)|^2 \) has a local max at \( s = 0 \), we have
\[
\frac{d}{ds} |\alpha(s)|^2 \bigg|_{s=0} = 0 \quad \text{and} \quad \frac{d^2}{ds^2} |\alpha(s)|^2 \bigg|_{s=0} \leq 0.
\]
These conditions imply
\[
\dot{\alpha}(0) \cdot \alpha(0) = 0 \quad \text{and} \quad |\dot{\alpha}(0)|^2 + \ddot{\alpha}(0) \cdot \alpha(0) \leq 0.
\]
Since \( \dot{\alpha} \) is a unit vector, the second inequality becomes
\[
\ddot{\alpha}(0) \cdot \alpha(0) \leq -1.
\]
This means
\[
|\ddot{\alpha}(0) \cdot \alpha(0)| \geq 1.
\]
Since \( k = |\ddot{\alpha}(0)| \) and \( |\alpha(0)| = 3 \), the Cauchy-Schwarz inequality implies
\[
3k = |\alpha(0)||\ddot{\alpha}(0)| \geq |\ddot{\alpha}(0) \cdot \alpha(0)| \geq 1.
\]
Dividing by 3 now gives the result.

2. (25 points) (3-3.2) A line passing through \( (0, 0, a) \in \mathbb{R}^3 \) is parameterized by
\[
\gamma(t) = (0, 0, a) + t(\cos a, \sin a, 0).
\]
Find a parameterization for the surface which is the union of all such lines.

**Solution:**
\[
X(u, v) = (u \cos v, u \sin v, v).
\]

3. (25 points) (3-2.8, Proposition 2, page 167) Let
\[
S_r = \{(x, y, x^2 + y^2) : x^2 + y^2 < r^2\}.
\]
1. Find an expression for the area of the image of the Gauss map of \( S_r \).
2. Find an expression for the area of $S_r$.
3. Denote the area of $S_r$ by $A(r)$ and the area of the Gauss image by $G(r)$. Compute
   \[
   \lim_{r \to 0} \frac{G(r)}{A(r)}.
   \]

**Solution:**

\[
A(r) = 2\pi \int_0^r \sqrt{1 + 4t^2} \, t \, dt = \frac{\pi}{6} [(1 + 4r^2)^{3/2} - 1].
\]

$G(r)$ is the area of a spherical cap on the unit sphere above the ball of radius $2r/\sqrt{1 + 4r^2}$. That is,

\[
G(r) = 2\pi \int_0^{\sqrt{1 + 4r^2}} \sqrt{1 + \frac{t^2}{1 - t^2}} \, t \, dt = 2\pi \left[1 - \frac{1}{\sqrt{1 + 4r^2}}\right].
\]

Thus,

\[
\frac{G(r)}{A(r)} = \frac{12}{12} \frac{1 - (1 + 4r^2)^{-1/2}}{(1 + 4r^2)^{3/2} - 1}.
\]

One application of L’Hopital’s rule gives that the desired limit of this expression is 4, which is the Gauss curvature of the paraboloid at the origin.

4. (25 points) (4-2.4,12) Find an isometry from
   \[B_1 = \{(x, y, 0) : x^2 + y^2 < 1\}\]
   to a surface $S$ of constant mean curvature $\pi/2$. Specify clearly the surface $S$ and the isometry.

**Solution:** Since $B_1$ is flat, $S$ must have Gauss curvature $K \equiv 0$. A cylinder is an obvious choice. In order to get $H = \pi/2$, we take

\[S = \{(x, y, z) : x^2 + y^2 = 1/\pi^2\}.
\]

Let $\phi : B_1 \to S$ by

\[\phi(x, y, 0) = \left(\frac{\cos(\pi x)}{\pi}, \frac{\sin(\pi x)}{\pi}, y\right).
\]

($\phi$ wraps the disk around the cylinder.) Observe that $\phi$ is one-to-one since the circumference of $S$ is the same as the diameter of $B_1$. Also, we have

\[\phi_x = (-\sin(\pi x), \cos(\pi x), 0) \quad \text{and} \quad \phi_y = (0, 0, 1).
\]
Thus, $|\phi_x| = 1$, $\phi_x \cdot \phi_y = 0$, and $|\phi_y| = 1$. Given $v = (a, b) \in T_p B_1$ and $w = (c, d) \in T_p (B_1)$, we have

$$d\phi(v) = \phi_x a + \phi_y b \quad \text{and} \quad d\phi(w) = \phi_x c + \phi_y d.$$ 

Thus,

$$\langle d\phi(v), d\phi(w) \rangle = \langle \phi_x a + \phi_y b, \phi_x c + \phi_y d \rangle$$

$$= ac + bd$$

$$= \langle v, w \rangle.$$ 

Thus, $\phi$ is an isometry.