1. A function $f : (0, 1) \to \mathbb{R}$ is called lower semicontinuous at $x_0$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

**Correction:** We should say that a function $f : A \to \mathbb{R}$ is called lower semicontinuous at $x_0$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x)$$

where $A$ is any subset of $\mathbb{R}$ and

$$\liminf_{x \to x_0} f(x) := \lim_{\delta \to 0} \inf_{0 < |x-x_0| < \delta} f(x).$$

Of course, in the limit, we only consider points $x \in A$. It can be an exercise to show this is the same as

$$\sup_{\delta > 0} \inf_{0 < |x-x_0| < \delta} f(x).$$

The function is lower semicontinuous if it is lower semicontinuous at every point. Similarly, $f$ is upper semicontinuous if

$$f(x_0) \geq \limsup_{x \to x_0} f(x).$$

**Correction:** For parts (a-b) below, we assume $A = (0, 1)$. For parts (c-f) we assume $A = [0, 1]$.

(a) Show that $f$ is continuous at a point if and only if it is both upper and lower semicontinuous at the point.

(b) Show that every lower semicontinuous function is Borel measurable.

(c) Show that a function is lower semicontinuous if and only if there is a sequence of lower semicontinuous step functions $\tilde{f}_j$ with $\tilde{f}_1 \leq \tilde{f}_2 \leq \tilde{f}_3 \leq \cdots$ and

$$\lim_{j \to \infty} \tilde{f}_j = f.$$

(A step function is one for which there are finitely many points $0 = x_0 < x_2 < x_3 < \cdots < x_k = 1$ such that $\tilde{f}$ is constant on each open interval $(x_j, x_{j+1})$.)

(d) Show that a function $f$ is lower semicontinuous if and only if there is an increasing sequence of continuous functions which converge to $f$.

(e) Show that a lower semicontinuous function which is bounded from below attains its minimum value on $[0, 1]$. 

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(f) Let \( f \) be a bounded function on \([0, 1]\) and set

\[
g(x) = \sup_{r>0} \inf_{|x|<r} f(\xi).
\]

Show that \( g \) is lower semicontinuous and that \( g \) is the largest lower semicontinuous function dominated by \( f \), i.e., if \( \tilde{f} \leq f \) is lower semicontinuous, then \( \tilde{f} \leq g \).

2. If \( f : \mathbb{R} \to \mathbb{R} \), then the set of points of continuity of \( f \) is a \( G_\delta \).

3. If \( f_1, f_2, f_3, \ldots \) is any sequence of continuous functions defined on \( \mathbb{R} \), then the set of points at which the sequence converges is an \( F_{\sigma\delta} \).

4. For \( A \subset \mathbb{R} \) define

\[
m^* A = \inf_{A \subset \bigcup [a_j, b_j]} \sum (b_j - a_j)
\]

where the infimum is taken over countable unions of closed intervals containing \( A \).

Show that \( m^* \) is an outer measure, and that

\[
m^* Z = 0 \implies m^*(A \cup Z) = m^* A
\]

for any set \( A \subset \mathbb{R} \).

5. Let \( f \) be a nonnegative Lebesgue measurable function on \([0, 1]\) which is bounded. Let \( \epsilon > 0 \).

(a) There is a nonnegative simple function \( \tilde{f} \leq f \) such that

\[
|f(x) - \tilde{f}(x)| < \epsilon
\]

for all \( x \).

(b) There is a nonnegative step function \( \bar{g} \leq \tilde{f} \) such that

\[
m\{x : \bar{g}(x) \neq \tilde{f}(x)\} < \epsilon/2.
\]

Hint: Use the fact that \( m \) is Borel regular.

Correction: Part (b) is impossible for \( f = \tilde{f} = \chi_{[0,1]} \setminus \mathbb{Q} \). Forget about \( \bar{g} \leq \tilde{f} \).

(c) There is a nonnegative continuous function \( \bar{f} \leq \bar{g} \) such that

\[
m\{x : \bar{f}(x) \neq \bar{g}(x)\} < \epsilon/2.
\]

(d) \( m\{x : \tilde{f}(x) \neq f(x)\} < \epsilon \).

Correction: Part (d) should read “\( m\{x : \tilde{f}(x) \neq f(x)\} < \epsilon \)” Actually, it’s true that you can find a continuous function which agrees with \( f \) except on a set of arbitrarily small measure. That’s called Lusin’s Theorem, but it’s a bit harder than what I had in mind here. A good idea for the final.
6. Let $f$ be a nonnegative measurable function on $\mathbb{R}$. Show that

$$\int f = 0 \implies f = 0 \text{ a.e.}$$

7. Let $\bar{B}_r(x_0) = \{x : |x - x_0| \leq r\}$ denote the closed ball of radius $r$ in $\mathbb{R}^n$ centered at $x_0$. Let $\mathcal{C}$ be a collection of such closed balls with bounded radii, i.e., there is some $R > 0$ such that $r < R$ for every $\bar{B}_r(x) \in \mathcal{C}$.

Show that there is a countable disjoint subcollection of the balls $\{\bar{B}_{r_j}(x_j)\}$ in $\mathcal{C}$ such that

$$\bigcup_{\mathcal{C}} \bar{B}_r(x) \subset \bigcup_{j} \bar{B}_{5r_j}(x_j).$$

Hint: Consider the biggest radii $R/2 \leq r < R$ first, and take a maximal collection of disjoint balls. Then consider the “next smaller radii” $R/4 \leq r < R/2$ etc..