Math 6341, Final Exam: Various Topics

Name: ____________________

1. (25 points) (3.5.9-10, Hamilton-Jacobi PDE)

(a) Define the convex dual (Legendre transform) of a function $H : \mathbb{R}^n \to \mathbb{R}$.

(b) Write down the Hopf-Lax formula associated with the IVP

$$\begin{cases}
  u_t + H(Du) = 0 \\
  u(x, 0) = u_0(x).
\end{cases}$$

(c) Give conditions on the Hamiltonian $H$ and the initial function $u_0$ under which your formula from part (b) provides a solution of the IVP. (Explain)

Solution:

(a) $$H^*(v) = \sup_p \{ p \cdot v - H(p) \}.$$  

(b) The Hopf-Lax formula is

$$u(x, t) = \min_{\xi \in \mathbb{R}^n} \left\{ tL \left( \frac{x - \xi}{t} \right) + u_0(\xi) \right\}$$

where $L = H^*$ is given in (a) above.

(c) In order for the Hopf-Lax formula to be well defined, we need that $L$ is convex satisfying

$$\lim_{|v| \to \infty} \frac{L(v)}{|v|} = +\infty$$

and $u_0$ Lipschitz continuous.

The conditions on $L$ will hold if $H$ is convex and satisfies

$$\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty.$$

The Hopf-Lax formula provides a weak solution for the IVP if we assume in addition that either $H$ is uniformly convex or $u_0$ is semi-concave.

Under either of these assumptions, the solution obtained will be unique.
2. (25 points) (Green’s Functions for Laplace’s PDE, §2.2.4) If $u, v \in C^2(\bar{\Omega})$ satisfy
\[
\begin{align*}
\Delta u &= f \\
\left. u \right|_{\partial \Omega} &= u_0,
\end{align*}
\]
and
\[
\begin{align*}
\Delta v &= 0 \\
v(\xi)\big|_{\xi \in \partial \Omega} &= \Phi(\xi - x)\big|_{\xi \in \partial \Omega},
\end{align*}
\]
where $\Phi$ is the fundamental solution of Laplace’s PDE, then find a formula expressing
\[
\int_{\xi \in \partial \Omega} \Phi(\xi - x) Du(\xi) \cdot n
\]
where $n$ is the outward normal to $\partial \Omega$ in terms of $f$, $u_0$, $v$, and $Dv$.

**Solution:** By the divergence theorem
\[
\int_{\Omega} (u \Delta v - v \Delta u) = \int_{\partial \Omega} (uDv \cdot n - vDu \cdot n).
\]
Substituting from the boundary value problems, this becomes
\[
-\int_{\Omega} vf = \int_{\partial \Omega} u_0 Dv \cdot n - \int_{\xi \in \partial \Omega} \Phi(\xi - x) Du(\xi) \cdot n.
\]
Thus,
\[
\int_{\xi \in \partial \Omega} \Phi(\xi - x) Du(\xi) \cdot n = \int_{\Omega} vf + \int_{\partial \Omega} u_0 Dv \cdot n.
\]
3. (25 points) (4.7.2) Find a separated variables solution of

\[
\begin{cases}
\Delta u = 0 \text{ on } \mathbb{R}^2 \\
u(x, 0) = 0, \quad u_y(x, 0) = \sin x.
\end{cases}
\]

Explain your reasoning carefully.

**Solution:** Setting \( u = f(x)g(y) \), we obtain away from \( f = 0 \) or \( g = 0 \) a separation constant \( \lambda \) such that

\[
-\frac{f''}{f} = \frac{g''}{g} = \lambda.
\]

Thus, we obtain two ODEs

\[
f'' = -\lambda f \quad \text{and} \quad g'' = \lambda g.
\]

The first boundary condition gives \( f(x)g(0) = 0 \) from which we conclude \( g(0) = 0 \), since \( f(x) = 0 \) cannot lead to a solution satisfying the other boundary condition.

The second boundary condition is \( f(x)g'(0) = \sin x \). Therefore,

\[
f(x) = \frac{\sin x}{g'(0)}.
\]

It follows from this that \( f'' = -f \). In view of the first ODE, we must have \( \lambda = 1 \). The second ODE with the first boundary condition then yields

\[
g(y) = g'(0) \sinh y.
\]

The solution is thus,

\[
u(x, y) = f(x)g(y) = \sinh y \sin x.
\]
4. (25 points) (Fourier Transform, §4.3.1)

(a) Give an example showing that \(L^1(\mathbb{R}^n)\) is not a subset of \(L^2(\mathbb{R}^n)\). Justify your assertion.

(b) Give an example showing that \(L^2(\mathbb{R}^n)\) is not a subset of \(L^1(\mathbb{R}^n)\). Justify your assertion.

(c) For \(u \in L^2(\mathbb{R}^n) \setminus L^1(\mathbb{R}^n)\), assume there are two sequences of functions \(u_j \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\) with \(|u_j - u|_{L^2} \to 0\) and \(\tilde{u}_j \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\) with \(|\tilde{u}_j - u|_{L^2} \to 0\). Using Plancharel’s Theorem, it can be shown that there are functions \(v\) and \(\tilde{v}\) both in \(L^2(\mathbb{R}^n)\) with

\[
|\hat{u}_j - v|_{L^2} \to 0 \quad \text{and} \quad |\hat{\tilde{u}}_j - \tilde{v}|_{L^2} \to 0.
\]

Show that \(v = \tilde{v}\).

Solution:

(a) 

\[
u(x) = \begin{cases} 
1/|x|^{n-1/2}, & 0 < |x| < 1 \\
0, & x = 0, \ |x| \geq 1 
\end{cases}
\]

has \(u \in L^1 \setminus L^2\).

\[
\int |u| = \int_0^1 \left( \int_{\partial B_r} \frac{1}{r^{n-1/2}} \right) \, dr 
= n \omega_n \int_0^1 r^{-1/2} \, dr 
= 2 n \omega_n r^{1/2} \bigg|_{r=0}^1 
= 2 n \omega_n 
< \infty.
\]

\[
\int |u|^2 = \int_0^1 \left( \int_{\partial B_r} \frac{1}{r^{2n-1}} \right) \, dr 
= n \omega_n \int_0^1 r^{-n} \, dr 
= n (1 - n) \omega_n r^{-n+1} \bigg|_{r=0}^1 
= +\infty.
\]

(b) 

\[
u(x) = \begin{cases} 
1/|x|^n, & |x| > 1 \\
0, & |x| \leq 1 
\end{cases}
\]
has \( u \in L^2 \setminus L^1 \).

\[
\int |u| = \int_1^\infty \left( \int_{\partial B_r} \frac{1}{r^n} \right) dr \\
= n \omega_n \int_1^\infty (1/r) dr \\
= n \omega_n \log(r)|_{r=1}^{\infty} \\
= \infty.
\]

\[
\int |u|^2 = \int_1^\infty \left( \int_{\partial B_r} \frac{1}{r^{2n}} \right) dr \\
= n \omega_n \int_1^\infty r^{-n-1} dr \\
= -\omega_n r^{-n}|_{r=1}^{\infty} \\
= \omega_n \\
< \infty.
\]

(c)

\[
|\bar{v} - v|_{L^2} = \lim_{j \to \infty} |\hat{u}_j - \hat{u}_j|_{L^2} \\
= \lim_{j \to \infty} |\hat{u}_j - u_j|_{L^2} \\
= \lim_{j \to \infty} |\bar{u}_j - u_j|_{L^2} \\
\leq \lim_{j \to \infty} |\bar{u}_j - u|_{L^2} + |u - u_j|_{L^2} \\
= 0.
\]

The first equality uses the continuity of the \( L^2 \) norm which can be further justified by the \( L^2 \) triangle inequality as follows

\[
||g|_{L^2} - |f|_{L^2}| \leq |g - f|_{L^2}.
\]

Thus, when \( g \) is close to \( f \) in \( L^2 \), the norms of \( g \) and \( f \) are also close to each other.

The third equality uses Plancharel’s Theorem.