1. (25 points) (3.5.3) Show that
\[ U(x, y, a, b) = \sqrt{1 - (x - a)^2 - (y - b)^2} \]
is a complete integral for the PDE
\[ u^2(|Du|^2 + 1) = 1. \]
Find the corresponding envelope solution.

**Solution:** The first requirement is that for each \( a \) and \( b \) fixed \( u(x, t) = U(x, y, a, b) \) solves the equation. In fact,
\[ u_x = -\frac{x - a}{\sqrt{1 - (x - a)^2 - (y - b)^2}} \quad \text{and} \quad u_y = -\frac{y - b}{\sqrt{1 - (x - a)^2 - (y - b)^2}}. \]
Thus,
\[ |Du|^2 + 1 = \frac{1}{1 - (x - a)^2 - (y - b)^2} = \frac{1}{u^2}. \]
Second, the 2 × 3 envelope matrix
\[
\begin{pmatrix}
\frac{x-a}{\sqrt{1-(x-a)^2-(y-b)^2}} & \frac{1-(y-b)^2}{(1-(x-a)^2-(y-b)^2)^{3/2}} & \frac{(x-a)(y-b)}{(1-(x-a)^2-(y-b)^2)^{3/2}} \\
\frac{y-b}{\sqrt{1-(x-a)^2-(y-b)^2}} & \frac{1-(x-a)^2}{(1-(x-a)^2-(y-b)^2)^{3/2}} & \frac{(x-a)(y-b)}{(1-(x-a)^2-(y-b)^2)^{3/2}}
\end{pmatrix}
\]
is required to have rank 2. Notice that the determinant of the 2 × 2 matrix obtained by deleting the last column is
\[
\frac{x - a - (x - a)^3 - (y - b) + (y - b)^3}{(1 - (x - a)^2 - (y - b)^2)^2}.
\]
This is nonzero unless \((x - a)[1 - (x - a)^2] = (y - b)[1 - (y - b)^2]\). Since for fixed \( x \) and \( y \), the set of \((a, b) \in \mathbb{R}^2\) satisfying this relation is a closed set, we obtain an open set (the complement) on which the envelope matrix has rank at least 2 (and it can’t have rank more than 2), so
\[ U(x, y, a, b) = \sqrt{1 - (x - a)^2 - (y - b)^2} \]
provides a complete integral on that set.

To obtain the envelope, we consider the system of equations \( U_a = 0 \) and \( U_b = 0 \) and attempt to solve for \( a \) and \( b \): These equations are
\[
\frac{x - a}{\sqrt{1 - (x - a)^2 - (y - b)^2}} = 0 \quad \text{and} \quad \frac{y - b}{\sqrt{1 - (x - a)^2 - (y - b)^2}} = 0.
\]
The solution is \( a = x \) and \( b = y \). Thus, the envelope is
\[ U(x, y, x, y) = 1. \]
2. (25 points) (3.5.5) Use the method of characteristics to solve
\[
\begin{align*}
\begin{array}{ll}
 xu_x + yu_y = 2u & \text{on } \mathbb{R}^2 \\
 u(x, 1) = x.
\end{array}
\end{align*}
\]
**Solution:** We set \( v(t) = u(\xi(t), \eta(t)) \) where \((\xi, \eta)\) parameterizes a curve passing through \((x, y)\) and satisfying \(\xi' = \xi, \eta' = \eta\).
In this way, we see that \(\xi = xe^t, \eta = ye^t\), and
\[
v' = xe^t u_x(xe^t, ye^t) + ye^t u_y(xe^t, ye^t) = 2v.
\]
It follows that \(v = v(0)e^{2t}\).
Next, for fixed \((x, y)\), we wish to find \(t\) such that \(\eta(t) = 1\) (in order to hit the Cauchy data curve). That is, we want \(ye^t = 1\) or \(e^t = 1/y\) (at least away from \(y = 0\)).
Substituting this choice into the expression for \(v\), we find
\[
u(x/y, 1) = u(x, y)/y^2.
\]
That is,
\[
u(x, y) = xy
\]
which is easily seen to be an entire solution of the PDE.

3. (25 points) (3.5.20) Find the characteristic curves and the solution of the IVP
\[
\begin{align*}
\begin{array}{ll}
 u_t + (u^2/2)_x = 0 & \text{on } \mathbb{R} \times (0, \infty) \\
 u(x, 0) = \begin{cases} 
 0, & x \leq 0 \\
 x, & 0 \leq x \leq 1 \\
 1, & 1 \leq x.
\end{cases}
\end{array}
\end{align*}
\]
**Solution:** We first attempt to find characteristic curves. Setting \(v = u(\xi, \tau)\), we want
\[
\xi' = v, \quad \tau' = 1,
\]
and
\[
v' = \xi' u_x(\xi, \tau) + \tau' u_t(\xi, \tau) = 0.
\]
Solving this system of ODEs, we get \(v = u(x_0)\), assuming we cross the Cauchy data curve \(t = 0\) at a point \(x_0\),
\[
\xi = u(x_0)t + x_0,
\]
and \(\tau = t\). Evidently, there are three cases:
1. \( x_0 \leq 0 \). Then the characteristics \( \xi \equiv x_0, \tau = t \) cover the region \( x \leq 0 \) and yield the solution
   \[ u \equiv 0, \quad x \leq 0. \]

2. \( 0 \leq x_0 \leq 1 \). Here the characteristics are
   \[ \xi = x_0t + x_0 = x_0(t + 1), \quad \tau = t. \]
   Such a characteristic hits the point \((x, t)\) if \( 0 \leq x \leq 1 + t \) and
   \[ x_0 = x/(1 + t). \]
   It follows that
   \[ u(x, t) = \frac{x}{1 + t}, \quad 0 \leq x \leq 1 + t. \]

3. The remaining characteristics with \( x_0 \geq 1 \) are given by \( \xi = t + x_0 \). These cover the region \( x \geq 1 + t \), and yield the solution \( u \equiv 1 \) there.

4. (25 points) (4.7.2) Find a separated variables solution of
   \[
   \begin{cases}
   \Delta u = 0 \text{ on } \mathbb{R}^2 \\
   u(x, 0) = 0, \quad u_y(x, 0) = \sin x.
   \end{cases}
   \]
   Explain your reasoning carefully.

**Solution:** Setting \( u = f(x)g(y) \), we obtain away from \( f = 0 \) or \( g = 0 \) a separation constant \( \lambda \) such that
   \[ \frac{-f''}{f} = \frac{g''}{g} = \lambda. \]
   Thus, we obtain two ODEs
   \[ f'' = -\lambda f \quad \text{and} \quad g'' = \lambda g. \]
   The first boundary condition gives \( f(x)g(0) = 0 \) from which we conclude \( g(0) = 0 \), since \( f(x) = 0 \) cannot lead to a solution satisfying the other boundary condition.
   The second boundary condition is \( f(x)g'(0) = \sin x \). Therefore,
   \[ f(x) = \frac{\sin x}{g'(0)}. \]
   It follows from this that \( f'' = -f \). In view of the first ODE, we must have \( \lambda = 1 \).
   The second ODE with the first boundary condition then yields
   \[ g(y) = g'(0) \sinh y. \]
   The solution is thus,
   \[ u(x, y) = f(x)g(y) = \sinh y \sin x. \]