1. (20 points) (4.8.16) Find the best linear approximation $\ell(x) = mx + b$ for the data $f(-1) = -2$, $f(0) = 0$, and $f(1) = 3$. (Hint: Use the least squares approximation method.)

**Solution:** A perfect fit would satisfy $-m + b = -2$, $b = 0$, and $m + b = 3$. That is,

$$
\begin{pmatrix}
-1 & 1 \\
0 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
m \\
b
\end{pmatrix} = \begin{pmatrix}
-2 \\
0 \\
3
\end{pmatrix}.
$$

This is not possible as one can see from row reduction of the coefficient matrix which gives

$$
\begin{pmatrix}
-1 & 1 & -2 \\
0 & 1 & 0 \\
0 & 2 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & 1 & -2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Thus, we seek the solution corresponding to projection of $(-2, 0, 3)^T$ onto the image. That is, we solve instead

$$
\begin{pmatrix}
-1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
m \\
b
\end{pmatrix} = \begin{pmatrix}
-1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 \\
0 \\
0
\end{pmatrix}.
$$

That is,

$$
\begin{pmatrix}
2 & 0 & 1 \\
0 & 3 & 0
\end{pmatrix}
\begin{pmatrix}
m \\
b
\end{pmatrix} = \begin{pmatrix}
5 \\
1
\end{pmatrix}.
$$

Thus, the best fit has $m = 5/2$ and $b = 1/3$. 

![Graph](image.png)
2. (20 points) (8.1.6) We model an evaporating substance with the assumption that the rate of evaporation is proportional to the exposed surface area. If a spherical volume evaporates so that its radius halves in six months, how long will it take for the volume to half?

**Solution:** We begin with the relation

\[
\frac{d}{dt} \left( \frac{4}{3} \pi r^3 \right) = -4\alpha \pi r^2.
\]

This tells us \( r' = -\alpha \) is constant. Consequently, \( r = -\alpha t + r_0 \), and \( -\alpha/2 + r_0 = r_0/2 \) (measuring time in years). Thus, \( \alpha = r_0 \) and \( r = r_0(1-t) \). The volume as a function of \( t \) is

\[
\frac{4}{3} r_0^3 (1-t)^3.
\]

We want to know when this quantity is \( 2r_0^3/3 \). That is, when \( (1-t)^3 = 1/2 \) or \( t = 1 - 1/\sqrt[3]{2} \approx 0.2 \) years, or about 2.5 months. Obviously, it will be less than 6 months. Why?
3. (20 points) (8.6.34) Solve the initial value problem
\[
\begin{align*}
y'' - 5y' + 6y &= 2e^x + 6x - 5 \\
y(0) &= 0 = y'(0).
\end{align*}
\]

**Solution:** The general solution of the associated homogeneous ODE is
\[
y_h(x) = ae^{2x} + be^{3x}.
\]

There is no interference between this solution and the forcing terms, so we can find a particular solution of \(u'' - 5u' + 6u = 2e^x\) which has the form \(u = \alpha e^x\). That solution is \(u = e^x\). Also, setting \(v = \alpha x + \beta\), we can solve \(v'' - 5v' + 6v = -5\alpha + 6\alpha x + 6\beta = 6x - 5\) with \(v = x\). Thus, a particular solution is \(y_p = u + v = e^x + x\). The general solution of the ODE is therefore
\[
y = ae^{2x} + be^{3x} + e^x + x
\]
where \(a\) and \(b\) are arbitrary constants. In order to get the initial conditions, we need \(a + b + 1 = 0\) and \(2a + 3b + 2 = 0\). That is, \(b = 0\) and \(a = -1\). Therefore, the solution of the initial value problem is
\[
y = -e^{2x} + e^x + x.
\]
4. (20 points) (8.6.34) Use the Laplace transform to solve the initial value problem
\[
\begin{align*}
\begin{cases}
y'' - 5y' + 6y &= 2e^x + 6x - 5 \\
y(0) = 0 &= y'(0).
\end{cases}
\end{align*}
\]

**Solution:** Since the initial conditions are all zero (homogeneous), the Laplace transform of the initial value problem is
\[
s^2Y - 5sY + 6Y = \frac{2}{s-1} + \frac{6}{s^2} - \frac{5}{s}.
\]
That is,
\[
Y = \frac{2}{(s-1)(s-2)(s-3)} + \frac{6}{s(s-2)(s-3)} - \frac{5}{s(s-2)(s-3)}.
\]
By partial fractions,
\[
\frac{1}{(s-2)(s-3)} = \frac{1}{s-3} - \frac{1}{s-2} = \mathcal{L}[e^{3x}] - \mathcal{L}[e^{2x}].
\]
Thus, we can use the “convolution” property of Laplace transform to get
\[
Y = 2\mathcal{L}\left[\int_0^x e^{x-\tau} e^{3\tau} d\tau\right] - 2\mathcal{L}\left[\int_0^x e^{x-\tau} e^{2\tau} d\tau\right] + 6\mathcal{L}\left[\int_0^x (x-\tau)e^{3\tau} d\tau\right] - 6\mathcal{L}\left[\int_0^x (x-\tau)e^{2\tau} d\tau\right] - 5\mathcal{L}\left[\int_0^x e^{3\tau} d\tau\right] + 5\mathcal{L}\left[\int_0^x e^{2\tau} d\tau\right].
\]
Evaluating the integrals, we get
\[
Y = 2\mathcal{L}\left[e^{x}(e^{2x} - 1)/2\right] - 2\mathcal{L}\left[e^{x}(e^{x} - 1)\right] + 6\mathcal{L}\left[x(e^{3x} - 1)/3\right] - 6\mathcal{L}\left[\int_0^x \tau e^{3\tau} d\tau\right] - 6\mathcal{L}\left[x(e^{2x} - 1)/2\right] + 6\mathcal{L}\left[\int_0^x \tau e^{2\tau} d\tau\right] - 5\mathcal{L}\left[(e^{3x} - 1)/3\right] + 5\mathcal{L}\left[(e^{2x} - 1)/2\right] = \mathcal{L}\left[2(e^{3x} - e^{x})/2 - 2(e^{2x} - e^{x})\right] + 6\mathcal{L}\left[x(e^{3x} - 1)/3\right] - 6\mathcal{L}\left[xe^{3x}/3 - (e^{3x} - 1)/9\right] - 6\mathcal{L}\left[x(e^{2x} - 1)/2\right] + 6\mathcal{L}\left[xe^{2x}/2 - (e^{2x} - 1)/4\right] - 5\mathcal{L}\left[(e^{3x} - 1)/3\right] + 5\mathcal{L}\left[(e^{2x} - 1)/2\right].
\]
Therefore,
\[
y = e^{3x} + e^{x} - 2e^{2x} + 2xe^{3x} - 2x - 2xe^{3x} + 2e^{3x}/3 - 2/3 - 3xe^{2x} + 3x + 3xe^{2x} - 3e^{2x}/2 + 3/2 - 5e^{3x}/3 + 5/3 + 5e^{2x}/2 - 5/2 = e^x - e^{2x} + x.
\]
5. (20 points) (8.11.9) A damped oscillator is modeled by the operator $L[y] = y'' + 2y' + 10y$, and is started in motion with the initial conditions $y(0) = 1$, $y'(0) = 0$. At some positive time $t_0$ an impulsive force stops the system so that $y(t) = 0$ for $t > t_0$. At what time can such an impulse be applied? Give also the direction and magnitude of the impulse. (Hint: Model the initial value problem with $L[y] = a\delta(t - t_0)$.)

**Solution:**

The Laplace transform of the problem described above is

$$(s^2 + 2s + 10)Y - s - 2 = ae^{-st_0}.$$ 

Since $s^2 + 2s + 10 = (s + 1)^2 + 9$,

$$Y = \frac{s}{(s + 1)^2 + 9} + \frac{2}{(s + 1)^2 + 9} + \frac{ae^{-st_0}}{(s + 1)^2 + 9}.$$ 

Shifting in $s$, we see

$$\mathcal{L}[e^{-t}\cos(3t)] = \frac{s + 1}{(s + 1)^2 + 9}, \quad \mathcal{L}[e^{-t}\sin(3t)] = \frac{3}{(s + 1)^2 + 9}.$$ 

Furthermore shifting in $s$ and $t$, we find

$$\mathcal{L}[e^{-(t-t_0)}\sin(3(t - t_0))\mathcal{H}(t-t_0)] = \frac{3e^{-st_0}}{(s + 1)^2 + 9}$$

where $\mathcal{H}$ is the Heaviside function. It follows that

$$Y = \mathcal{L}[e^{-t}\cos(3t)] + \frac{1}{3}\mathcal{L}[e^{-t}\sin(3t)] + \frac{ae^{t_0}}{3}\mathcal{L}[e^{-t}\sin(3(t - t_0))\mathcal{H}(t-t_0)].$$

Therefore,

$$y = \frac{e^{-t}}{3}[3\cos(3t) + \sin(3t) + ae^{t_0}\sin(3(t - t_0))\mathcal{H}(t-t_0)]$$

$$= \frac{e^{-t}}{3}[\sqrt{10}\cos(3t - \psi) + ae^{t_0}\sin(3(t - t_0))\mathcal{H}(t-t_0)]$$

where $\cos \psi = 3/\sqrt{10}$ and $\sin \psi = 1/\sqrt{10}$, that is, $\psi = \sin^{-1}(1/\sqrt{10})$. Evidently, the impulse must come at a time when the undisturbed motion passes through equilibrium. This means when $3t - \psi = \pi/2 + \pi k$ for some $k = 0, 1, 2, \ldots$. Thus, setting $t_0 = (\psi + \pi/2 + \pi k)/3$, we desire for $t > t_0$ to have

$$\sqrt{10}\cos(3t - \psi) + ae^{t_0}\sin(3(t - t_0))\mathcal{H}(t-t_0) = 0.$$ 

Since $\sin(3t - \psi - \pi/2 + \pi k) = -\cos(3t - \psi + \pi k) = (-1)^{k+1}\cos(3t - \psi)$, we want

$$\sqrt{10} + (-1)^{k+1}ae^{t_0} = 0.$$
or the magnitude can be
\[ a = (-1)^k \sqrt{10} e^{-t_0} \]
at time \( t_0 = (\psi + \pi/2 + \pi k)/3 \), for some \( k = 0, 1, 2, \ldots \).

The first such time would be \( t_0 = (\sin^{-1}(1/\sqrt{10}) + \pi/2)/3 \), and we would need a positive impulse \( a = \sqrt{10} e^{-t_0} \) to stop the system. This makes sense since the system is released with positive displacement and will be moving down on the first pass through equilibrium.