The evaluation of barrier option prices under stochastic volatility

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\textbf{A B S T R A C T}

This paper considers the problem of numerically evaluating barrier option prices when the dynamics of the underlying are driven by stochastic volatility following the square root process of Heston (1993) [7]. We develop a method of lines approach to evaluate the price as well as the delta and gamma of the option. The method is able to efficiently handle both continuously monitored and discretely monitored barrier options and can also handle barrier options with early exercise features. In the latter case, we can calculate the early exercise boundary of an American barrier option in both the continuously and discretely monitored cases.

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1. Introduction

Barrier options are path-dependent options and are very popular in foreign exchange markets. They have a payoff that is dependent on the realized asset path via its level; certain aspects of the contract are triggered if the asset price becomes too high or too low during the option’s life. For example, an up-and-out call option pays off the usual max(\(S - K, 0\)) at expiry unless at any time during the life of the option the underlying asset has traded at a value \(H\) or higher. In this example, if the asset reaches this level (from below, obviously) then it is said to “knock out” and become worthless. Apart from “out” options like this, there are also “in” options which only receive a payoff if a certain level is reached, otherwise they expire worthless. Barrier options are popular for a number of reasons. The purchaser can use them to hedge very specific cash flows with similar properties. Usually, the purchaser has very precise views about the direction of the market. If he or she wants the payoff from a call option but does not want to pay for all the upside potential, believing that the upward movement of the underlying will be limited prior to expiry, then he may choose to buy an up-and-out call. It will be cheaper than a similar vanilla call, since the upside is severely limited. If he is right and the barrier is not triggered he gets the payoff he wanted. The closer that the barrier is to the current asset price then the greater the likelihood of the option being knocked out, and thus the cheaper the contract.

Barrier options are common path-dependent options traded in the financial markets. The derivation of the pricing formula for barrier options was pioneered by Merton [1] in his seminal paper on option pricing. A list of pricing formulas for one-asset barrier options and multi-asset barrier options both under the geometric Brownian motion (GBM) framework can be found in the articles by Rich [2] and Wong and Kwok [3], respectively. Gao et al. [4] analyzed option contracts with both knock-out barrier and American early exercise features. Zvan et al. [5] have discussed the oscillatory behavior of the
Crank–Nicolson method for pricing barrier options, and they applied the backward Euler method in order to avoid unwanted oscillations.

Derivative securities are commonly written on underlying assets with return dynamics that are not sufficiently well described by the GBM process proposed by Black and Scholes [6]. There have been numerous efforts to develop alternative asset return models that are capable of capturing the leptokurtic features found in financial market data, and subsequently to use these models to develop option prices that better reflect the volatility smiles and skews found in market traded options. One of the classical ways to develop option pricing models that are capable of generating such behavior is to allow the volatility to evolve stochastically, for instance according to the square-root process introduced by Heston [7]. The evaluation of barrier option prices under the Heston stochastic volatility model has been extensively discussed by Griebsch [8] in her thesis.

However, there are certain drawbacks in the evaluation of the Barrier option prices under SV using either tree or finite difference methods, these include the fact that the convergence is rather slow and it takes more effort to obtain accurate hedge ratios. Yousef [9,10] have developed a higher order smoothing scheme for pricing barrier options under stochastic volatility. The method is stable and converges rapidly which overcome some drawbacks of the finite difference methods. But those papers do not discuss how to handle the possible early exercise features of the barrier option pricing problem.

It turns out that another well known method, the method of lines is able to overcome those disadvantages. In this paper, we introduce a unifying approach to price both continuously and discretely monitored barrier options without or with early exercise features. Specifically, except for American style knock-in options,\(^1\) we are able to price all other kinds of European or American barrier options using the framework developed here.

The remainder of the paper is structured as follows. Section 2 outlines the problem of both continuously and discretely monitored barrier options where the underlying asset follows stochastic volatility dynamics. In Section 3 we outline the basic idea of the method of lines approach and implement it to find the price profile of the barrier option. A number of numerical examples that demonstrate the computational advantages of the method of lines approach are provided in Section 4. Finally we discuss the impact of stochastic volatility on the prices of the barrier option in Section 5 before we draw some conclusions in Section 6.

2. Problem statement-barrier option with stochastic volatility

Let \(C(S, v, \tau)\) denote the price of an up-and-out (UO) call option with time to maturity \(\tau\) written on a stock of price \(S\) and variance \(v\) that pays a continuously compounded dividend yield \(q\). The option has strike price \(K\) and a barrier \(H\).

Analogously to the setting in [7], the dynamics for the share price \(S\) under the risk neutral measure are governed by the stochastic differential equation (SDE) system\(^2\)

\[
\begin{align*}
    dS &= (r - q)S dt + \sqrt{v} S dZ_1, \\
    dv &= \kappa_v (\theta - v) dt + \sigma \sqrt{v} dZ_2,
\end{align*}
\]

where \(Z_1, Z_2\) are standard Wiener processes and \(\mathbb{E}(dZ_1 dZ_2) = \rho dt\) with \(\mathbb{E}\) the expectation operator under a particular risk neutral measure. In (1), \(r\) is the risk free rate of interest. In (2) the parameter \(\sigma\) is the so called vol-of-vol (in fact, \(\sigma^2 v\) is the variance of the variance process \(v\)). The parameters \(\kappa_v\) and \(\theta\) are respectively the rate of mean reversion and long run variance of the process for the variance \(v\). These are under the risk-neutral measure and are related to the corresponding quantities under the physical measure by a parameter that appears in the market price of volatility risk.\(^3\)

We are also able to write down the above system (1)–(2) using independent Wiener processes. Let \(W_1 = Z_2\) and \(Z_1 = \rho W_1 + \sqrt{1 - \rho^2} W_2\) where \(W_1\) and \(W_2\) are independent Wiener processes under the risk neutral measure. Then,

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\[\text{1 Strictly speaking, American style knock-in options could be priced numerically as well. But the approach will be more complicated than that indicated in this paper. In fact, let us take an American up-and-in option } C_{\text{u}}(S, v, \tau, H) \text{ as an example. If } H \text{ is the upper barrier, then we have}\]

\[
\begin{align*}
    C_{\text{u}}(S, v, \tau, H) &= \int_0^\infty \int_0^\tau C(H, v, S) \mathcal{P}(H, v, \tau | S) d\tau dv, \\
    \mathcal{P}(H, v, \tau | S) &= \int_0^\tau \mathcal{P}\left(H, v, \tau - \xi \mid S, v\right) \mathcal{P}(H, v, \tau | S, v) d\xi dv;
\end{align*}
\]

where \(\mathcal{P}(H, v, \tau | S, v)\) is a standard American option with stock price \(H\), variance \(v\) and time to maturity \(\tau\) and \(\mathcal{P}(H, v, \tau | S)\) is the transition density (Green's function) of the two dimensional processes \((S, v)\). Hence, we could price \(C(H, v, \tau | S, v)\) using the method of lines for certain quadrature points on \(v_1\) and \(\xi\). But then we would need to work out the value of the Greens function \(\mathcal{P}(H, v, \tau | S, v)\) on the corresponding quadrature points as well and then evaluate the two dimensional numerical integral maybe using the sparse grid approach. Thus, it is hard to implement the detailed approach in this paper to price American-style knock in options directly.\]

\[\text{2 Note that } \tau = T - t, \text{ where } T \text{ is the maturity date of the option and } t \text{ is the running time.}\]

\[\text{3 Of course, since we are using a numerical technique we could in fact use more general processes for } S \text{ and } v. \text{ The choice of the Heston processes is driven partly by that this has become a very traditional stochastic volatility model and partly because a companion paper on the evaluation of European compound options under stochastic volatility uses techniques based on a knowledge of the characteristic function for the stochastic volatility process, which is known for the Heston process (see [11]), and can be used for comparison purposes.}\]

\[\text{4 In fact, if it is assumed that the market price of risk associated with the uncertainty driving the variance process has the form } \lambda \sqrt{v}, \text{ where } \lambda \text{ is a constant (this was the assumption in [7]) and } \kappa_v^0, \theta \text{ are the corresponding parameters under the physical measure, then } \kappa_v = \kappa_v^0 + \lambda \sigma, \theta = \frac{\kappa_v^0}{\lambda^2}. \text{ In this formulation, the choice of a risk neutral measure comes down to deciding the parameters. This could for instance be done by a calibration procedure.}\]
the dynamics of $S$ and $v$ can be rewritten in terms of independent Wiener processes as

$$
\begin{align*}
    dS &= (r - q)Sdt + \sqrt{v}SdW_1 + \sqrt{1 - \rho^2}dW_2, \quad (3) \\
    dv &= \kappa_v(\theta_v - v)dt + \sigma\sqrt{v}dW_1. \quad (4)
\end{align*}
$$

The price of a barrier option under stochastic volatility at time to maturity $\tau$, $C(S, v, \tau)$, can be formulated as the solution to a partial differential equation (PDE) problem. We need to solve the PDE for the value of the barrier option $C(S, v, \tau)$ given by

$$
\frac{\partial C}{\partial \tau} = \mathcal{K}C - rC, \quad (5)
$$
on the interval $0 \leq \tau \leq T$, where the Kolmogorov operator $\mathcal{K}$ is given by

$$
\mathcal{K} = \frac{\nu S^2}{2} \frac{\partial^2}{\partial S^2} + \rho \sigma v \frac{\partial^2}{\partial S \partial v} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial v^2} + (r - q)S \frac{\partial}{\partial S} + (\kappa_v(\theta_v - v) - \lambda v) \frac{\partial}{\partial v}, \quad (6)
$$

where $\lambda$ is the constant appearing in the equation for the market price of volatility risk, which as stated in Footnote 4 is of the form $\lambda\sqrt{v}$.

Both the terminal and boundary conditions need to be specified depending on the detailed specifications of the barrier options.

Note that the Fichera function for the time discretized pricing equation (5) is

$$
h(S, v, \tau) = [(r - q)S - (vS + \sigma \rho S/2)]n_1 + [\kappa(\theta - v) - \lambda v - (\sigma \rho v/2 + \sigma^2/2)]n_2.
$$

On $S = 0$ we have $(n_1, n_2) = (1, 0)$ and $h(0, v, \tau) = 0$. This means the pricing equation (5) has to hold for all parameters. We note that $C = 0$ is the solution of this equation. However on $v = 0$ we have $(n_1, n_2) = (0, 1)$ so that

$$
h(S, 0, \tau) = \kappa \theta - \sigma^2/2.
$$

The Fichera theory says that if $h(S, 0, \tau) < 0$ then one CAN impose a Dirichlet condition on $v = 0$. However, one can also impose lots of other boundary conditions. In particular, one can require that the pricing equation defines a Venttsel boundary condition for which the Eq. (5) has a unique solution. Hence we can always solve the pricing equation (5) at $v = 0$ with $C(0, 0, \tau)$ as boundary condition regardless of the Feller condition. This makes good sense because the problem does not change in any way when the parameters are perturbed slightly but the Fichera function changes sign. Both at $S = 0$ and $v = 0$ the pricing equation is the natural boundary condition for which the solution can be expected to be continuous with respect to the parameters in the equation.

In the following discussions, we assume the Feller condition holds in each one of the following cases. To obtain a consistent approximation of Eq. (5) near $v = 0$, we fit a quadratic polynomial through the option prices in the neighborhood of $v_0$, say, $v_1$, $v_2$ and $v_3$, and then use it to extrapolate a value for $v_0$. This quadratic extrapolation also insures that the differential equation (5) holds with $v = 0$.

2.1. Continuously monitored barrier option

A continuously monitored barrier option is an option which is monitored all the time between the current time and the maturity of the option at time $T$. The option with or without early exercise features, has the terminal condition

$$
C(S, v, 0) = (S - K)^+. \quad (7)
$$

The domain for the up and out call option is

$$
0 < S < H, \quad 0 < v < \infty, \quad 0 < \tau < T. \quad (8)
$$

The boundary conditions for the barrier option without the early exercise features are:

$$
\begin{align*}
    C(0, v, \tau) &= 0; \quad (9) \\
    C(H, v, \tau) &= 0; \quad (10) \\
    \lim_{v \to \infty} C_v(S, v, \tau) &= 0. \quad (11)
\end{align*}
$$

The option with early exercise features has the free (early exercise) boundary condition

$$
C(b(v, \tau), v, \tau) = b(v, \tau) - K, \quad \text{when } b(v, \tau) < H \quad (12)
$$

where $S = b(v, \tau)$ is the early exercise boundary for the barrier option at time to maturity $\tau$ and variance $v$, and there also hold the smooth-pasting conditions

$$
\lim_{S \to b(v, \tau)} \frac{\partial C}{\partial S} = 1, \quad \lim_{S \to b(v, \tau)} \frac{\partial C}{\partial v} = 0. \quad (13)
$$
In the above case,
\[ C(S, v, \tau) = S - K, \quad \forall b(v, \tau) < S < H. \]

However, if we cannot find a \( b(v, \tau) < H \) then 
\[ C(H, v, \tau) = H - K, \]

because technically, for the knock-out event and the exercise date to be well defined, the option contract is defined in a way such that when the asset price first touches the barrier, the option holder has the option to either exercise or let the option be knocked out. Since we assume the rebate is equal to zero, the option should be exercised once the asset price touches the barrier.

2.2. Discretely monitored barrier option

A discretely monitored barrier option is an option which is monitored only at discrete dates \( t \leq t_1 < t_2 < \cdots < t_N \leq T \), while the option is not monitored at other times. The option with or without early exercise features, has the terminal condition
\[ C(S, v, 0) = (S - K)^+. \]  

(14)
The domain for the up and out call option is:
\[ S \in \left\{ (0, H), \quad \tau \in \{ T - t_N, T - t_{N-1}, \ldots, T - t_1 \}, \right\} \]
and
\[ 0 < v < \infty, \quad 0 < \tau < T. \]

The boundary conditions for the barrier option without early exercise features are:
\[ C(0, v, \tau) = 0; \]  
\[ C(H, v, \tau) = 0, \quad \forall \tau \in \{ T - t_N, \ldots, T - t_1 \}; \]  
\[ \lim_{S \to \infty} C(S, v, \tau) = 0, \quad \forall \tau \notin \{ T - t_N, \ldots, T - t_1 \}; \]  
\[ \lim_{v \to \infty} C_v(S, v, \tau) = 0. \]  

(15) 
(16) 
(17) 
(18)

A discretely monitored barrier option with the early exercise feature, at the monitoring times \( \tau \in \{ T - t_N, \ldots, T - t_1 \} \), has the free (early exercise) boundary condition
\[ C(b(v, \tau), v, \tau) = b(v, \tau) - K, \quad \text{when } b(v, \tau) < H \]  
\[ \]  
(19)

where \( b(v, \tau) \) is the early exercise boundary for the barrier option at time to maturity \( \tau \) and variance \( v \), and the smooth-pasting conditions
\[ \lim_{S \to b(v, \tau)} \frac{\partial C}{\partial S} = 1, \quad \lim_{S \to b(v, \tau)} \frac{\partial C}{\partial v} = 0. \]  

(20)

In the above case, we have 
\[ C(S, v, \tau) = S - K, \quad \forall b(v, \tau) < S < H \]
so that \( C(S, v, \tau) \) is known over \( 0 < S < H \). If there is no such \( b(v, \tau) \) then for the same reason as the case for the continuously monitored option, \( C(S, v, \tau) \) must satisfy
\[ C(H, v, \tau) = H - K. \]

At all other times \( \tau \notin \{ T - t_N, \ldots, T - t_1 \} \), standard American option free boundary conditions apply.

Before going into details of the valuation, the following relations between the payoffs of barrier options and vanilla options are pointed out. The in–out parity for European barrier options, namely
\[ \text{knock-in + knock-out} = \text{vanilla}; \]
allows us to consider only the family of knock-out options for the valuation using the method of lines (MOL) since we are able to price vanilla options under Heston model using the analytic solution from Heston [7]. In the next section, we are going to discuss the details of computing the up-and-out barrier option prices by implementing the MOL.

3. Method of lines (MOL) approach

In this section, we will provide the details of the implementation of the Method of Lines for Eq. (5) on the computational domain
\[ 0 < S_0 < S < H, \quad 0 < v < v_{\text{max}}, \quad 0 < \tau < T \]
for continuously monitored barrier options and
\[ 0 < S_0 < S < H, \quad \forall \tau \in \{ T - t_1, \ldots, T - t_1 \}; \]
\[ 0 < S_0 < S < S_{\text{max}}, \quad \forall \tau \notin \{ T - t_1, \ldots, T - t_1 \}; \]
\[ 0 < v < v_{\text{max}}, \quad 0 < \tau < T, \]
for discretely monitored barrier option.

The key idea behind the method of lines is to approximate a PDE with a system of ordinary differential equations (ODEs), the solution of which can be obtained with ODE techniques. When volatility is constant, the system of ODEs is obtained by discretizing time. For the PDE (5), we must in addition discretize the variance, \( \sigma^2 \). Typically we will set the maximum variance to be \( \sigma^2_{\text{max}} = 100\% \). Furthermore, we discretize the time to expiry according to \( \tau_n = n \Delta \tau \), where \( n = 0, 1, 2, \ldots, N \) and \( \tau_N = T \). We denote the option price along the variance line \( v_m \) and time line \( \tau_n \) by \( C(S, v_m, \tau_n) = C_m^n(S) \), and set
\[
V(S, v_m, \tau_n) \equiv \frac{\partial C(S, v_m, \tau_n)}{\partial S} = V_m^n(S),
\]
which is of course the option delta at the particular grid point.

We now select finite difference approximations for the derivative terms with respect to \( v \). For the second order term, at the grid point \((S, v_m, \tau_n)\) we use the standard central difference scheme
\[
\frac{\partial^2 C}{\partial v^2} = \frac{C_{m+1}^n - 2C_m^n + C_{m-1}^n}{(\Delta v)^2}.
\]
Similarly for the cross-derivatives terms at the grid point \((S, v_m, \tau_n)\), we use the central difference approximation
\[
\frac{\partial^2 C}{\partial S \partial v} = \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta v}.
\]
Since the coefficients of the second order derivative terms go to zero as \( v \to 0 \), we use an upwinding finite difference scheme for the first order derivative term (see [12]), such that, at the grid point \((S, v_m, \tau_n)\) we have
\[
\frac{\partial C}{\partial v} = \begin{cases} 
\frac{C_{m+1}^n - C_m^n}{\Delta v} & \text{if } v \leq \frac{\alpha}{\beta}, \\
\frac{C_m^n - C_{m-1}^n}{\Delta v} & \text{if } v > \frac{\alpha}{\beta},
\end{cases}
\]
where \( \alpha = \kappa_v \theta_v \) and \( \beta = \kappa_v + \lambda_v \). If the coefficients of the \( v \)-derivatives, especially close to \( v = 0 \), do not have diagonal dominance then the maximum principle does not apply to the discrete equations and oscillatory solutions might arise. Hence upwinding helps to stabilize the finite difference scheme with respect to \( v \).

Next we must select a discretization for the time derivative. Initially we use a standard backward difference scheme for 3 time steps, given at the grid point \((S, v_m, \tau_n)\) by
\[
\frac{\partial C}{\partial \tau} = \frac{C_m^n - C_{m-1}^{n-1}}{\Delta \tau}.
\]
This approximation is only first order accurate with respect to time. For the case of the standard American put option, it is known from Meyer [13] that the accuracy of the method of lines increases considerably by using a second order approximation for the time derivative, specifically
\[
\frac{\partial C}{\partial \tau} = \frac{3C_m^n - C_{m-1}^{n-1} - 1C_m^{n-2}}{2\Delta \tau} - \frac{1C_m^{n-1} - C_m^{n-2}}{2\Delta \tau}.
\]
Thus we initiate the method of lines solution by using (25) for the first several time steps, and then switch to (26) for all subsequent time steps. For a discretely monitored barrier option that we will discuss below, we switch back to the backward difference scheme (25) for three time steps right after each monitoring time and then switch to (26) before the next monitoring time.

Applying (22)–(26) to the PDE (5), we now need to solve a system of second order ODEs at each time step and variance grid point. For the first few time steps, the ODE system at the grid point \( v = v_m \) and \( \tau = \tau_n \) is
\[
\frac{v_m \sigma^2}{2} \left( \frac{C_m^n}{S^2} + \rho \sigma v_m \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta v} + \frac{\sigma^2 v_m C_{m+1}^n - 2C_m^n + C_{m-1}^n}{(\Delta v)^2} + \frac{\alpha - \beta v_m}{2} \frac{C_{m+1}^n - 2C_m^n + C_{m-1}^n}{\Delta v} + (r - q)S \frac{dC_m^n}{dS} - rC_m^n - \frac{C_m^n - C_{m-1}^{n-1}}{\Delta \tau} = 0, \right)
\]
and for all subsequent time steps the ODE system has the form

$$\frac{v_n}{2} \frac{d^2 C_n}{dS^2} + \rho \sigma v_n S \frac{dC_n}{dS} \frac{V_{n+1}^m - V_n^m}{2\Delta z} + \frac{\sigma^2 v_n C_m^{n+1} - 2 C_m^n + C_{m-1}^n}{(\Delta v)^2} + \frac{\alpha - \beta v_n C_m^{n+1} - C_{m-1}^n}{\Delta v} + \frac{\alpha - \beta v_n C_m^n - C_{m-1}^n}{\Delta v} = 0. \quad (28)$$

We require two boundary conditions in the $v$ direction, one at $v_0$ and the other at $v_M$. For large values of $v$, the rate of change of the option price with respect to $v$ converges to zero. So for sufficiently large values of $v$, one can treat this rate of change as zero without any impact on the accuracy of the solution at other values of $v$. Thus we set $dc/dv = (v_{M+1} - v_{M-1})/(2\Delta v) = 0$ and solve (27) also for $m = M$. This makes the boundary condition approximation second order. In this case we have a system of $M$ equations along the variance boundary $v = v_M$. To obtain a consistent approximation of Eq. (5) near $v = 0$ we fit a quadratic polynomial through the option prices at $v_1$, $v_2$, and $v_3$, and then use it to extrapolate a value for $v_0$ which then is used in (27) and (28) for $m = 1$. It turns out that this provides a satisfactory estimate of the price along $v_0$ for the purpose of generating a stable solution for small values of $v$.\(^5\)

After taking the boundary conditions into consideration, we must solve a system of $M - 1$ second order ODEs in $S$ along the line segment $(S, v_m, t_m)$, $S \in [S_0, H]$ or $S \in [S_0, S_{\text{max}}]$ depending on the properties of the barrier option for $m = 1, \ldots, M - 1$ and fixed $t_m$. We solve this system of ODEs iteratively for increasing values of $m$, using the latest available estimates for $C_m^{n+1}$, $C_{m-1}^{n+1}$, $V_m^{n+1}$, and $V_{m-1}^{n+1}$. The initial estimates for $C_m^n$ and $V_m^n$ are simply $C_{m-1}^{n-1}$ and $V_{m-1}^{n-1}$, then we use the latest estimates for $C_m^n$ and $V_m^n$ found during the current iteration through the variance lines. At a grid value of $S$ we continue to cycle through the lines until the change in the price between successive iterations falls below a tolerance of $10^{-8}$. We accept the last iterate as the solution $C_m^n(S)$ and proceed to the next time level.

The system of ODEs (27) and (28), after rearrangement, maybe cast into the generic first order system form

$$\frac{dc^n}{dS} = V^n_m,$$  \hspace{1cm} (29)

$$\frac{dv^n}{dS} = A_m(S)C^n_m + B_m(S)V^n_m + P^n_m(S),$$  \hspace{1cm} (30)

where, for example, for Eq. (27)

$$A_m(S) = \frac{2}{v_m}\left[\frac{\sigma^2 v_m}{\Delta v^2} + \frac{\alpha - \beta v_m}{\Delta v} + r + \frac{1}{\Delta \tau}\right], \quad B_m(S) = \frac{2(1 - q)S}{v_m\Delta v^2},$$

and where $P^n_m(S)$ is a function of $C_m^{n+1}$, $C_{m-1}^{n+1}$, $V_m^{n+1}$, $V_{m-1}^{n+1}$, $C_m^{n-1}$ and $C_m^{n-2}$ that may be inferred from the RHS of (27) or (28).

The restriction to the line segment $0 < S_0 < S < H$ or $0 < S_0 < S < S_{\text{max}}$ assures that the coefficients of (30) remain bounded. We solve the system (29)-(30) using the Riccati transform, full details of which are provided by Meyer [13].\(^6\) Note that we are only able to apply the Riccati transform to the system (29)-(30) provided that both equations are treated as ODEs. We use an iterative technique in which the price ($C_m^n$) and the derivative ($V_m^n$) terms are updated until the price profile converges.

The Riccati transformation is given by

$$C_m^n(S) = R_m(S)V_m^n(S) + W_m^n(S),$$  \hspace{1cm} (31)

where $R$ and $W$ are solutions to the initial value problems

$$\frac{dR_m}{dS} = 1 - B_m(S)R_m(S) - A_m(S)(R_m(S))^2, \quad R_m(S_0) = 0,$$  \hspace{1cm} (32)

$$\frac{dW_m^n}{dS} = -A_m(S)R_m(S)W_m^n - R_m(S)P_m^n(S), \quad W_m^n(S_0) = 0.$$  \hspace{1cm} (33)

Note that the coefficients in (32) depend on whether (27) or (28) applies, but for each case and for a constant time step, Eq. (32) is independent of time and needs to be computed only once for each $m$.

To obtain $V_m^n(S)$ required for (31) we need to solve the ODE (34),\(^7\)

$$\frac{dv^n}{dS} = A_m(S)(R_m(S)V_m^n + W_m^n(S)) + B_m(S)V_m^n + P_m^n(S).$$  \hspace{1cm} (34)

---

\(^5\) See [14] for more discussion and justification for this procedure for handling the boundary conditions at $v = 0$ for stochastic volatility models.

\(^6\) Chapter 2 of Meyer [13] has the most detailed description of the method. The integration of the differential equations with the trapezoidal rule is sensible because the method is second order and therefore consistent with the difference quotients used in the $v$ and $t$ direction, except for the upwinding term. Its advantage is easy communication with the solution being generated from the preceding time level. An adaptive integrator in $S$ would require interpolation of functions stored only at the mesh points.

\(^7\) It is certainly true that the grid should be selectively refined. Our code does that in the $S$ direction when we solve Eqs. (32)-(34), although not adaptively. We have more points near the strike and near the barrier. However, we have not systematically studied the convergence of the iteration when we have un-evenly spaced lines. Our sense is, however, that the number of iterations required for convergence depends on the smallest distance between lines, not on the total number of lines. The number of points along lines has a direct influence on run-times but does not influence the number of iterations required.
subject to an initial condition which depends on the properties and the specifications of the barrier options:

- For continuously monitored barrier options without early exercise opportunities, we solve (33) for increasing values of $S$, ranging from $S_0 < S < H$. Using the fact that $C^n_m(H) = 0$ we obtain from (31) the terminal condition

$$V^n_m(H) = - \frac{W^n_m(H)}{R^n_m(H)}$$

(35)

and then integrate (34) from $S = H$ to $S = S_0$.

- For continuously monitored barrier options with early exercise opportunity we integrate (32), (33) from $S_0$ to $S_{max}$ and monitor the function

$$\phi(S) = R^n_m(S) + W^n_m(S) - (S - K).$$

If $\phi(S^*) = 0$ for some $S^* \in (S_0, H)$ then $S^*$ is the early exercise boundary $b^n_m = b^n_m$ at the grid point $(v_m, t_n)$. In general $\phi(S)$ will change sign at most once on $[S_0, H]$, $b^n_0$ will change during the iteration but will converge as the prices converge. Once $b^n_0$ is found we integrate (34) backward from $b^n_0$ toward $S_0$ subject to the initial condition

$$V(b^n_0) = 1.$$

If $\phi(S)$ has no zero in $[S_0, H)$ then there is no early exercise below the barrier and we solve (34) subject to

$$V^n_m(H) = \frac{H - K - W^n_m(H)}{R^n_m(H)}.$$

In fact, at any time to maturity $\tau$, if the asset hits the barrier $H$, then the option will be exercised, namely, $C(H, v, \tau) = H - K$, according to the Riccati transform (31) we have

$$C^n_m(H) = R^n_m(H)V^n_m(H) + W^n_m(H) = H - K.$$

- For discretely monitored barrier options without early exercise features, the procedures to solve the PDE are similar to those for the continuously monitored counterpart, but we should change back to standard Euler backward time difference for 3 steps after each monitoring time and then switch to the second order scheme until the next monitoring time. The time difference in the Riccati equation should be adjusted in a similar manner as well.

- For discretely monitored options with early exercise features, we solve $R^n_m$ from the Riccati equation (32) and solve $W^n_m$ from the forward sweep (33) as usual. We find the free boundary point $S^*$ in the standard way as for the continuously monitored option but let $b^n_m = \min[S^*, H]$ at each of the monitoring dates and update the corresponding option value as well. At the non-monitoring dates, we set $b^n_0 = S^*$ as the early exercise boundary value which is used as the terminal value from which to work backward to solve Eq. (34) from $S = b^n_m$ to $S = S_0$. In this case, we still need to change back to the standard Euler backward time difference for 3 time steps after each monitoring time and then switch to the second order scheme before the next monitoring time. The time difference in the Riccati equation should be adjusted in a similar manner.

In Fig. 1 we illustrate one sweep through the grid points on the $v - \tau$ plane. In Fig. 2 we show the stencil for the typical grid point in Fig. 1, this essentially shows the grid point values of $C$ that enter the right-hand side of (30). Fig. 3 then illustrates the solution of (33) along a line in the $S$ direction from a typical grid point in the $v - \tau$ plane.

---

8 Zvan et al. [5] applied the backward Euler method in order to avoid unwanted oscillations in the Crank–Nicolson scheme. Here if the barrier condition holds then the delta is discontinuous at the barrier, so we need to restart the time evaluation over the next period from a discontinuous initial delta. For the same reason as Zvan et al. [5] a backward Euler method is applied here for the first few time steps.
There is no proven rate of convergence for the above algorithm. Its performance must be read off the tables of numbers in our examples in Section 4. There is an analysis of the MOL line iteration for an elliptic free boundary model problem in [15] but this problem does not contain a cross derivative term. For the time discrete elliptic problem the iteration at least for the uncorrelated case of $\rho = 0$ can be inferred to converge. For $|\rho| > 0$ convergence is only observed. Stability of the implicit two-level time discretization scheme is known for the heat equation and observed in our case. For a more comprehensive analysis and demonstration of the convergence of MOL in option pricing context, for instance efficiency plots, we refer the reader to Chiarella et al. [14].

4. Numerical examples

To demonstrate the performance of the method of lines outlined in Section 3 we implement the method for a given set of parameter values shown in Table 1,\(^9\) chosen to be consistent with the stochastic volatility parameters being used by

\(^9\) Here we assume a continuous dividend yield, however in the case of discrete dividends the computation would have to be restarted with a shifted initial value. That is straightforward since the stochastic volatility terms do not change at the ex-dividend time.
Table 1
Parameter values used for the barrier option. The stochastic volatility (SV) parameters are those used in Heston’s original paper.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>SV parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>0.03</td>
<td>( \theta )</td>
<td>0.1</td>
</tr>
<tr>
<td>( q )</td>
<td>0.05</td>
<td>( \kappa )</td>
<td>2.00</td>
</tr>
<tr>
<td>( T )</td>
<td>0.5</td>
<td>( \sigma )</td>
<td>0.1</td>
</tr>
<tr>
<td>( K )</td>
<td>100</td>
<td>( \lambda )</td>
<td>0.00</td>
</tr>
<tr>
<td>( \rho )</td>
<td>±0.50</td>
<td>( H )</td>
<td>130</td>
</tr>
</tbody>
</table>

Table 2
Prices of the continuously monitored barrier option without early exercise features computed using method of lines (MOL), finite difference (FD) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with \( \rho = -0.50 \) and \( v = 0.1 \).

<table>
<thead>
<tr>
<th>Method (N, M, ( S_{pts} ))</th>
<th>( S )</th>
<th>Runtime (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOL (50, 100, 1140)</td>
<td>80.9045</td>
<td>1.8807</td>
</tr>
<tr>
<td>MOL (100, 200, 6400)</td>
<td>0.9044</td>
<td>1.8781</td>
</tr>
<tr>
<td>FD (200, 100, 200)</td>
<td>0.9029</td>
<td>1.8778</td>
</tr>
<tr>
<td>MC (4000, 20, 1,000,000)</td>
<td>0.9046</td>
<td>1.8806</td>
</tr>
<tr>
<td>MC upper bound</td>
<td>0.9076</td>
<td>1.8849</td>
</tr>
<tr>
<td>MC lower bound</td>
<td>0.9017</td>
<td>1.8764</td>
</tr>
</tbody>
</table>

Table 3
Prices of the continuously monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with \( \rho = -0.50 \) and \( v = 0.1 \).

<table>
<thead>
<tr>
<th>Method (N, M, ( S_{pts} ))</th>
<th>( S )</th>
<th>Runtime (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOL (50, 100, 1140)</td>
<td>1.4009</td>
<td>3.9350</td>
</tr>
<tr>
<td>MOL (100, 200, 2440)</td>
<td>1.4012</td>
<td>3.9364</td>
</tr>
<tr>
<td>MOL (100, 200, 6400)</td>
<td>1.4012</td>
<td>3.9363</td>
</tr>
<tr>
<td>MOL (200, 400, 9100)</td>
<td>1.4015</td>
<td>3.9371</td>
</tr>
<tr>
<td>MC (100, 20, 50, 1,000,000)</td>
<td>1.3994</td>
<td>3.9238</td>
</tr>
<tr>
<td>MC upper bound</td>
<td>1.4058</td>
<td>3.9347</td>
</tr>
<tr>
<td>MC lower bound</td>
<td>1.3930</td>
<td>3.9129</td>
</tr>
</tbody>
</table>

Heston [7] and which have been standard in many papers undertaking numerical studies of stochastic volatility models.\(^{10}\) We use weekly monitoring frequency for the discretely monitored options.

The number of iterations is an important concept for the MOL, based on the computational experience, we found that at each time step the prices profile will converge to \( 10^{-8} \) after a maximum of 12–14 iterations independent of the type of the options.

Also in order to check and benchmark the results and to demonstrate the performance of the MOL, we use several available methods, such as the Finite Difference (FD) method (see [16]), Fourier Cosine Expansion (COS) method\(^{11}\) (see [18,19]) together with the Monte Carlo Simulation method (see [20]) to work out the prices of different kinds of barrier options to compare the prices from the MOL. Here we have chosen those methods as they are the best ones in calculating the prices and hedge ratios with respect to different barrier options with different features as the benchmark or alternative approach.

From Tables 2–5 we can see that the MOL is very efficient in producing the barrier option prices and it is also important to note that the MOL produces hedge ratios, such as deltas, gammas to the same level of accuracy as the prices themselves. In fact, delta and gamma are available from the differential equations. Delta \( (V^n_m) \) is the solution of ODE (34) but Gamma is calculated from the right hand side of Eq. (34) which is a direct calculation and does not require numerical differentiation. The iterative scheme will not stop running until the price profile converges to a certain accuracy, however based on the Riccati transform equation (31) the convergence of delta \( (V^n_m) \) should be faster than that of the price. The advantage of

\(^{10}\) The source code for all methods was implemented using NAG Fortran with the IMSL library running on the UTS, Faculty of Business F&E HPC Linux Cluster which consists of 8 nodes running Red Hat Enterprise Linux 4.0 (64 bit) with \( 2 \times 3.33 \) GHz, \( 2 \times 6 \) MB cache Quad Core Xeon X5470 Processors with 1333MHz FSB 8 GB DDR2-667 RAM.

\(^{11}\) We employed a variant of the COS method mainly to cater for the Heston model. This version of COS method has been presented in [17]. The lower efficiency of the COS method applied to pricing barrier options under the Heston model is mainly because we have to consider not only the transition of the spot price but also the transition of the stochastic variance. Hence it becomes truly a two-dimensional pricing problem that is dealt with in [17] by a combination of a Fourier Cosine series expansion, as in [18,19], and high-order quadrature rules in the other dimension. The numerical results in [17] demonstrate that the run time to price one single barrier option would be about \( 50–60 \) s. Our results are roughly comparable to theirs since the run time in Table 4 is for pricing 5 different options.
Table 4
Prices of the discretely monitored barrier option without early exercise features computed using method of lines (MOL), Fourier Cosine expansion (COS) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

<table>
<thead>
<tr>
<th>$\rho = -0.50$, $v = 0.1$</th>
<th>$S$ (Runtime (s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method ($N, M, S_{fin}$)</td>
<td>80 90 100 110 120</td>
</tr>
<tr>
<td>MOL (50, 100, 1140)</td>
<td>1.0764 2.5173 4.0895 4.9894 4.8291</td>
</tr>
<tr>
<td>MOL (100, 200, 6400)</td>
<td>1.0807 2.5289 4.1116 5.0235 4.8706</td>
</tr>
<tr>
<td>COS (100, 200, 100)</td>
<td>1.0809 2.4871 4.0454 4.9779 4.8646</td>
</tr>
<tr>
<td>MC (400, 20, 1,000,000)</td>
<td>1.0780 2.5257 4.1033 5.0166 4.8605</td>
</tr>
<tr>
<td>MC upper bound</td>
<td>1.0834 2.5339 4.1135 5.0279 4.8718</td>
</tr>
<tr>
<td>MC lower bound</td>
<td>1.0726 2.5175 4.0930 5.0054 4.8492</td>
</tr>
</tbody>
</table>

Table 5
Prices of the discretely monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

<table>
<thead>
<tr>
<th>$\rho = -0.50$, $v = 0.1$</th>
<th>$S$ (Runtime (s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method ($N, M, S_{fin}$)</td>
<td>80 90 100 110 120</td>
</tr>
<tr>
<td>MOL (50, 100, 1140)</td>
<td>1.4008 3.9339 8.3010 14.4446 22.0389</td>
</tr>
<tr>
<td>MOL (100, 250, 2400)</td>
<td>1.4012 3.9364 8.3025 14.4182 21.8719</td>
</tr>
<tr>
<td>MOL (150, 250, 6400)</td>
<td>1.4014 3.9368 8.3028 14.4157 21.8615</td>
</tr>
<tr>
<td>MC (100, 20, 50, 1,000,000)</td>
<td>1.4002 3.9338 8.2967 14.4285 21.9274</td>
</tr>
</tbody>
</table>

Fig. 4. Price profile of a continuously monitored up-and-out call option without early exercise opportunities.

the MOL is that the discontinuity of the gamma at the exercise boundary does not enter into the calculation along the line since we do not use difference quotients in $S$ straddling the boundary. The $C_{vv}$ term does straddle the early exercise boundary, but our numerical experiments indicate that it is better to base that difference approximation on the intrinsic value beyond the exercise boundary rather than some smooth (maybe quadratic) extrapolation of $C$ beyond the free boundary. It can also be seen from the efficiency plots in [14] that delta and gamma can achieve the same accuracy as the prices.

Figs. 4–11 demonstrate that the MOL is able to produce both smooth option prices, early exercise boundaries and option deltas which are a part of the solution we have after running the MOL.

In fact, Tables 2–5 show that

- the prices of continuously monitored European up-and-call option produced from the MOL are close to those prices generated from the finite difference method but the MOL provides better hedge ratios;
- the prices of discretely monitored European up-and-call option produced from the MOL are close to those prices generated from the Fourier Cosine Expansion method but the MOL is more efficient since the runtime of COS method shown in Table 4 are the time to produce only 5 prices while the runtime of the MOL is the time to have prices of all grid points;
Fig. 5. Price profile of a discretely monitored up-and-out call option without early exercise opportunities.

Fig. 6. Early exercise boundary of a continuously monitored up-and-out call option.

Fig. 7. Early exercise boundary of a discretely monitored up-and-out call option (including 3 monitoring dates).
• the prices of both continuously and discretely monitored American up-and-call option produced from the MOL are close to those prices generated from the Monte Carlo simulation\textsuperscript{12} which ran considerably longer than the MOL.

5. Impact of stochastic volatility on the prices of the barrier option

In this section, we explore the impact of stochastic volatility on the price profiles of Barrier options with various features. We consider two models for the underlying asset price: (i) the geometric Brownian motion (GBM) model of Black and Scholes [6] and Merton [1]; (ii) the stochastic volatility (SV) model of Heston [7]. Here we aim to observe the impact that stochastic volatility has on the shape of the price profile, where the variance of $S$ is consistent for both models.

Setting the spot variance to $v = 0.1$ (corresponding to a volatility – standard deviation – of 33%) in the SV model, we determine the time-averaged variance $s^2$ for $\ln S$ over the life of the option by using the characteristic function for the marginal density of $x = \ln S$ given in [21].

By requiring that $s^2$ be equal for both the models, we then determine the necessary parameter volatility $\sigma$ for the GBM to ensure that they both have consistent variance over the time period of interest. To match the time-averaged variance for the GBM and SV models for a 6-month option, the global volatilities, $s$, are 31.48% for $\rho = 0.50$, and 31.80% for $\rho = -0.50$. The value of $v$ in the SV model is 10%. Hence, the constant volatility $\sigma$ in GBM is chosen to be 31.48% for $\rho = 0.50$, and 31.80% for $\rho = -0.50$ in all the following comparisons.

\textsuperscript{12} The specification of each Monte Carlo simulation in the tables are the numbers in the parenthesis after MC which mean (No. of time steps, No. of volatility levels, No. of simulations) for the options without early exercise opportunities and (No. of time steps, No. of volatility levels, No. of early exercise opportunities, No. of simulations) for the options with early exercise opportunities, respectively.
Fig. 10. Delta profile of a discrete monitoring up-and-out call option without early exercise opportunities.

Fig. 11. Delta profile of a discrete monitoring up-and-out call option with early exercise opportunities.

Fig. 12. The effect of stochastic volatility on continuously monitored European up-and-out call (UOC) option. The correlation is $\rho = -0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 2.4197.

Figs. 12–15 demonstrates the prices differences of different types of barrier options under Heston stochastic volatility model and those option prices under the standard Geometric Brownian Motion.
Fig. 13. The effect of stochastic volatility on discretely monitored European up-and-out call option. The correlation is \( \rho = -0.5 \) and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 3.9487.

Fig. 14. The effect of stochastic volatility on the continuously monitored American up-and-out call option. The correlation is \( \rho = -0.5 \) and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 8.2917.

Fig. 15. The effect of stochastic volatility on discretely monitored American up-and-out call option. The correlation is \( \rho = -0.5 \) and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 8.3125.
6. Conclusion

We have studied the pricing of Barrier options under stochastic volatility using the Method of Lines. We also provide the Barrier option pricing results from Finite Difference method, Fourier Cosine Expansion method and Monte Carlo Simulation approach as benchmarks to the MOL.

It turns out that the MOL is able to handle both continuously and discretely monitored options with or without early exercise opportunities. Hence we believe this provides a unified framework to efficiently price various kinds of Barrier options with different kinds of properties. One main advantage of the MOL is that it produces the hedge ratios of the option, namely the deltas and gammas, to the same accuracy as the prices themselves within the same time frame.

In future research, the knock-in option under stochastic volatility with early exercise features should be further investigated.

References