Parabolic PDEs with hysteresis and quasivariational inequalities

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1. Introduction

In this paper, the parabolic equation

\[ u_t - \Delta u + w = 0 \quad \text{in} \quad Q_T := \Omega \times (0, T), \quad 0 < T < +\infty, \]  

subject to initial and boundary conditions, is considered with source term \( w = w(x, t) \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( 1 \leq N \leq 3 \), with smooth boundary \( \Gamma := \partial \Omega \). We are interested in Eq. (1.1), when the source \( w \) is given by hysteresis operator \( F(\cdot; w_0) \), namely

\[ w(x, t) := [F(u(\cdot, x); w_0(x))(t), \quad (x, t) \in Q_T, \]  

where \( w_0 \) being prescribed as an initial output of \( w \); its input–output relation is illustrated in Fig. 1.1 (see [1, Part I, III-2] for the precise definition); in the figure \( f_a \) and \( f_d \) are continuous and nondecreasing functions on \( \mathbb{R} \) such that \( f_a \leq f_d \) on \( \mathbb{R} \). We refer for some results on this sort of model to [1–4].

As is well known [1, Part I, III-2], (1.2) is equivalent to the following variational inequality

\[ w_t(x, t) + \partial I_{u(x, t)}(w(x, t)) \geq 0, \quad (x, t) \in Q_T, \]  

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where $I_{u(x,t)}(\cdot)$ is the indicator function of the interval $[f_u(u(x,t)), f_d(u(x,t))]$ and $\partial I_{u(x,t)}$ is its subdifferential in $\mathbb{R}$.

Equation (1.1) with a hysteresis source (1.2) has been studied as a simplified model, for instance, for a heat conduction system with source $w$ controlled by a distributed thermostat or for a reaction–diffusion system of the autocatalytic type, neglecting diffusion effect for $w$ (see [5]).

In this paper, taking account of diffusion effect in the input–output relation (1.3), we consider (1.1) coupled with

$$w_t - \kappa \Delta w + \partial I_u(w) \geq 0 \quad \text{in} \quad Q_T \tag{1.4}$$

where $\kappa$ is a (small) positive constant. This system $\{(1.1), (1.4)\}$ might be more realistic than $\{(1.1), (1.3)\}$ as a model for some kind of reaction–diffusion processes. Also, it can be regarded as regular approximation for $\{(1.1), (1.2)\}$ with approximation parameter $\kappa$. In fact, (1.3) is an ODE, while (1.4) is a PDE, so the function $w$ governed by (1.3) is not necessarily smooth in $x \in \Omega$, but the one by (1.4) might be smooth in $x \in \Omega$ on account of diffusion term $-\kappa \Delta w$. In expression (1.4), the subdifferential $\partial I_u(\cdot)$ depends on the unknown $u$, so system $\{(1.1), (1.4)\}$ is called a quasivariational inequality (see [6]).

As possible boundary conditions we consider

$$u = 0, \quad w = 0 \quad \text{on} \quad \Sigma_T := \Gamma \times (0, T) \quad \text{(Dirichlet type)} \tag{1.5}$$

or

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Sigma_T \quad \text{(Neumann type)}, \tag{1.6}$$

where $n$ is the unit outward normal vector to $\Gamma$. Of course, the nonhomogeneous Dirichlet boundary condition is able to be considered, but for simplicity we treat only the homogeneous case in this paper.
In this paper we shall try to show that
(i) the initial-boundary value problem mentioned above has a solution and it is unique;
(ii) the solution obtained in (i) converges to that of the problem with $\kappa = 0$ in a reasonable sense as $\kappa$ tends to 0.

The results obtained in this paper are completely new.

Our method is based on the time-dependent subdifferential technics in Hilbert spaces, which has been evolved to solve parabolic PDEs with time-dependent double obstacles, and it gives existence of a solution for a large class of initial data. The idea for uniqueness proof is to use extensively the continuous dependence of solutions upon obstacle functions for parabolic variational inequalities with time-dependent double obstacles (see [6]).

2. Existence and uniqueness results

Throughout this paper, let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $1 \leq N \leq 3$, with smooth boundary $\Gamma := \partial \Omega$, and put

$$Q_T := \Omega \times (0,T), \quad \Sigma_T := \Gamma \times (0,T) \quad \text{for finite } T > 0.$$ 

We use the following notation:

$$a(v,z) := \int_\Omega \nabla v \cdot \nabla z \, dx, \quad v,z \in H^1(\Omega); \quad (v,z) := \int_\Omega vz \, dx, \quad v,z \in L^2(\Omega).$$

Let $f_a$ and $f_d$ be functions from $\mathbb{R}$ into itself such that

\[
\begin{cases}
    f_a', f_a'', f_d', f_d'' \text{ are bounded on } \mathbb{R}, \\
    f_a \leq f_d, \quad f_a' \geq 0, \quad f_d' \geq 0 \quad \text{on } \mathbb{R};
\end{cases}
\]

hence $f_a$ and $f_d$ are functions in $W^{2,\infty}(\mathbb{R})$, and they are nondecreasing and Lipschitz continuous on $\mathbb{R}$. Now, for any given $u \in L^2(\Omega)$ we denote by $I_u(\cdot)$ the function on $L^2(\Omega)$ defined by

$$I_u(w) := \begin{cases} 0 & \text{if } w \in L^2(\Omega), \ f_a(u) \leq w \leq f_d(u) \text{ a.e. on } \Omega, \\
+\infty & \text{otherwise}. \end{cases}$$

Clearly, $I_u$ is proper, l.s.c. and convex on $L^2(\Omega)$, and the subdifferential $\partial I_u$ is a multifunction in $L^2(\Omega)$ which is defined as follows: $\xi \in \partial I_u(w)$ if and only if $w \in L^2(\Omega)$ with $f_a(u) \leq w \leq f_d(u)$ a.e. on $\Omega$ and $\xi \in L^2(\Omega)$ such that

\[
(\xi,z-w) \leq 0 \quad \text{for all } z \in L^2(\Omega) \text{ with } f_a(u) \leq z \leq f_d(u) \text{ a.e. on } \Omega; \quad (2.2)
\]

the above variational inequality (2.2) is equivalent to

\[
\begin{cases}
    \xi(x) \geq 0 & \text{for a.e. } x \text{ with } w(x) = f_d(u(x)), \\
    \xi(x) = 0 & \text{for a.e. } x \text{ with } f_a(u(x)) < w(x) < f_d(u(x)), \\
    \xi(x) \leq 0 & \text{for a.e. } x \text{ with } w(x) = f_a(u(x)).
\end{cases}
\]
Now, for a given constant \( \kappa \in (0, \kappa_0] \) (\( \kappa_0 \) is a fixed positive number) and initial data \( u_0, w_0 \) which satisfy

\[
\begin{align*}
\left\{ \begin{array}{l}
u_0 \in H^1_0(\Omega) \cap C(\overline{\Omega}), \quad w_0 \in C(\overline{\Omega}) \quad \text{with} \quad w_0 = 0 \quad \text{on} \quad \Gamma, \\
\text{(resp.} \quad u_0 \in H^1(\Omega) \cap C(\overline{\Omega}), \quad w_0 \in C(\overline{\Omega})), \\
\end{array} \right.
\end{align*}
\]

(2.3)

consider the following system \( P_{k, D}(u_0, w_0) \) (resp. \( P_{k, N}(u_0, w_0) \)):

\[
\begin{align*}
\left\{ \begin{array}{l}
u_t - \Delta u + w = 0 \quad \text{in} \quad Q_T, \\
\nu_t - \kappa \Delta w + \partial I(u)(w) \geq 0 \quad \text{in} \quad Q_T, \\
\nu = 0, \quad w = 0 \quad \text{(resp.} \quad \frac{\partial u}{\partial n} = 0, \quad \frac{\partial w}{\partial n} = 0) \quad \text{on} \quad \Sigma_T, \\
\nu(\cdot, 0) = u_0, \quad w(\cdot, 0) = w_0 \quad \text{in} \quad \Omega.
\end{array} \right.
\end{align*}
\]

We begin with the precise formulation of our problems.

**Definition 2.1.** A couple of functions \( \{u, w\} \) is called a (weak) solution of \( P_{k, D} \) (resp. \( P_{k, N} \)) on \( [0, T] \), \( 0 < T < +\infty \), if the following conditions (v1)--(v3) are satisfied:

(v1) \( u \in C(Q_T) \cap W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \)

\( w \in C(Q_T) \cap W^{1,2}_{\text{loc}}((0, T], L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^\infty_{\text{loc}}((0, T], H^1(\Omega)) \cap L^2_{\text{loc}}((0, T]; H^2(\Omega)). \)

(v2) \( u'(t) - \Delta u(t) + w(t) = 0 \) in \( L^2(\Omega) \) for a.e. \( t \in (0, T] \), where \( u' = \frac{\partial u}{\partial t} \) and

\[
u(t) = 0 \quad \text{(resp.} \quad \frac{\partial u(t)}{\partial n} = 0) \quad \text{on} \quad \Gamma \quad \text{(in the sense of traces) for a.e.} \quad t \in [0, T].
\]

(v3) There exists a function \( \xi \in L^2_{\text{loc}}((0, T]; L^2(\Omega)) \) such that

\[
v'(t) - \kappa \Delta w(t) + \xi(t) = 0 \quad \text{in} \quad L^2(\Omega) \quad \text{for a.e.} \quad t \in [0, T],
\]

\( \xi(t) \in \partial I(u)(w(t)) \quad \text{for a.e.} \quad t \in [0, T] \)

and

\[
\omega(t) = 0 \quad \text{(resp.} \quad \frac{\partial w(t)}{\partial n} = 0) \quad \text{on} \quad \Gamma \quad \text{(in the sense of traces) for a.e.} \quad t \in [0, T].
\]

Also, given initial data \( u_0, w_0 \in C(\overline{\Omega}), \) a couple of functions \( \{u, w\} \) is called a (weak) solution of the Cauchy problem \( P_{k, D}(u_0, w_0) \) (resp. \( P_{k, N}(u_0, w_0) \)), if it is a solution of \( P_{k, D} \) (resp. \( P_{k, N} \)) and \( u(0) = u_0, \quad w(0) = w_0. \)

**Remark 2.1.** By fundamental properties of linear heat equations the (weak) solution \( \{u, w\} \) of \( P_{k, D} \) (resp. \( P_{k, N} \)) automatically satisfies that

\[
u \in C([0, T]; H^1(\Omega)) \cap C_{\text{loc}}((0, T]; H^2(\Omega)) \cap C_{\text{loc}}^1((0, T]; L^2(\Omega)),
\]

\( w \in C_{\text{loc}}((0, T]; H^1(\Omega)). \)
The first result, which guarantees the uniqueness of the solution, is concerned with the continuous dependence of the solution on the initial data.

**Theorem 2.1.** Let \( \{u_0, w_0\}, i = 1, 2, \) be two sets of initial data and \( \{u_i, w_i\} \) be a solution of \( P_{k,D}(u_0, w_0) \) (resp. \( P_{k,N}(u_0, w_0) \)) on \( [0, T] \), \( 0 < T < +\infty \), for \( i = 1, 2 \). Then there is a positive constant \( A(T) \), depending on \( T \) but not on initial data, such that

\[
|u_1 - u_2|_{C([0, T]; \mathbb{R}^d)} + |w_1 - w_2|_{C([0, T]; \mathbb{R}^d)} \leq A(T)(|u_{01} - u_{02}|_{C([0, T])} + |w_{01} - w_{02}|_{C([0, T])}) \quad (2.4)
\]

**Remark 2.2.** More precisely, in Theorem 2.1, the constant \( A(T) \) depends on not only \( T \) but also the Lipschitz constants of \( f_a \) and \( f_d \).

Our existence result is stated as follows; in the rest of this paper we always assume that condition (2.1) is satisfied, and for problem \( P_{k,D} \) the following compatibility condition (2.5) is satisfied as well:

\[
f_a(0) \leq 0 \leq f_d(0). \quad (2.5)
\]

**Theorem 2.2.** Let \( \kappa \in (0, \kappa_0] \). Assume the initial data \( u_0, w_0 \) satisfy (2.3). Then, for every finite \( T > 0 \) problem \( P_{k,D}(u_0, w_0) \) (resp. \( P_{k,N}(u_0, w_0) \)) admits a solution \( \{u, w\} \) on \( [0, T] \) such that

\[
\begin{align*}
\{ \ & t^x u' \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad t^x w \in L^\infty(0, T; H^1(\Omega)), \\
& t^x w' \in L^2(0, T; L^2(\Omega)), \quad t^x \xi \in L^2(0, T; L^2(\Omega)),
\end{align*}
\]

where \( \xi \) is the function in (v3) of Definition 2.1. Moreover, in addition to (2.3), if \( u_0 \in H^2(\Omega), \quad w_0 \in H^1(\Omega), \quad (2.7) \)

then

\[
\begin{align*}
\{ \ & u \in C([0, T]; H^2(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \\
& u' \in L^2(0, T; H^1(\Omega)) \quad (\text{resp. } u' \in L^2(0, T; H^1(\Omega))), \\
& w \in W^{1,2}(0, T; L^2(\Omega)) \cap C([0, T]; H^1(\Omega)).
\end{align*}
\]

The solution \( \{u, w\} \) satisfying regularity (2.8) is called a strong solution of \( P_{k,D} \) (resp. \( P_{k,N} \)).

Using Theorem 2.2 together with some energy inequalities, we prove a theorem on the asymptotic convergence with respect to \( \kappa \).

**Theorem 2.3.** Assume that the initial data \( \{u_0, w_0\} \) satisfies (2.3) and (2.7). Let \( \{u_\kappa, w_\kappa\} \) be the strong solution of \( P_{k,D}(u_0, w_0) \) (resp. \( P_{k,N}(u_0, w_0) \)) on \( [0, T] \), \( 0 < T < +\infty \), for each \( \kappa \in (0, \kappa_0] \). Then \( \{u_\kappa, w_\kappa\} \) converges to a couple of functions \( \{u, w\} \), as \( \kappa \to 0 \), in the sense that

\[
\begin{align*}
u_\kappa(t) & \to u(t) \text{ weakly in } H^2(\Omega) \text{ and uniformly in } t \in [0, T], \\
u'_\kappa & \to u' \text{ in } C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
w_\kappa & \to w \text{ weakly in } W^{1,2}(0, T; L^2(\Omega)).
\end{align*}
\]

(2.9)
Moreover, the limit \{u, w\} is a unique solution of the initial-boundary value problem, denoted by \(P_D(u_0, w_0)\) (resp. \(P_N(u_0, w_0)\)):

\[
\begin{aligned}
\begin{cases}
    u'(t) - \Delta u(t) + w(t) = 0 &\text{in } L^2(\Omega) \text{ for a.e. } t \in [0, T], \\
    u(t) = 0 &\text{on } \Gamma \text{ for a.e. } t \in [0, T], \\
    w(x, t) = [F(u(x, \cdot); w_0(x))(t) &\text{for all } t \in [0, T] \text{ and a.e. } x \in \Omega,} \\
    u(0) = u_0, \quad w(0) = w_0
\end{cases}
\end{aligned}
\]  

(2.10)

Regarding problem (2.10), we refer to [1–4] for the existence and uniqueness of a solution; as was mentioned in the Introduction, the relation

\[ w(x, t) = [F(u(x, \cdot); w_0(x))(t) \text{ for all } t \in [0, T] \text{ and a.e. } x \in \Omega \]

is equivalent to

\[ w'(t) + \partial_{\partial D}^-(w(t)) \geq 0 \text{ in } L^2(\Omega) \text{ for a.e. } t \in [0, T]. \]

The proofs of Theorems 2.1, 2.2 and 2.3 will be given in Sections 3, 5 and 6, respectively.

3. Continuous dependence of solutions upon initial data

We recall an important lemma.

**Lemma 3.1.** Let \(\kappa \in (0, \kappa_0]\) and \(\{u_0, w_0\}_i, i = 1, 2, \) be two sets of initial data. Let \(\{u_i, w_i\}_i, i = 1, 2, \) be solutions of \(P_D(u_0, w_0)\) (resp. \(P_N(u_0, w_0)\)) on \( [0, T], 0 < T < +\infty. \) Then:

(i) For all \(0 \leq t \leq s \leq T, \)

\[
|u_1(t) - u_2(t)|_{C(\overline{\Omega})} \leq |u_01 - u_02|_{C(\overline{\Omega})} + \int_0^t |w_1(\tau) - w_2(\tau)|_{L^\infty(\Omega)} \, d\tau. \quad (3.1)
\]

(ii) For all \(0 \leq t \leq s \leq T, \)

\[
|w_1(t) - w_2(t)|_{C(\overline{\Omega})} \leq \max\{ |w_01 - w_02|_{C(\overline{\Omega})}, L_1 |u_1 - u_2|_{C(\overline{\Omega})}\}, \quad (3.2)
\]

where \(L_1\) is any common Lipschitz constant of \(f_a\) and \(f_d.\)

**Proof.** Inequality (3.1) is a direct consequence of [2; Lemma 3]. We now show (3.2). Putting \(\psi^1_a := f_a(u_1)\) and \(\psi^2_a := f_a(u_2),\) we observe from conditions (v1)–(v3) of Definition 2.1 for \(\{u_i, w_i\}_i, i = 1, 2,\) that \(w_i\) is the (unique) solution of the double obstacle problem

\[
\begin{aligned}
\begin{cases}
    w_i(t) \in K_i(t) &\text{for all } t \in [0, T], \\
    (w_i(t), w_i(t) - z) + \kappa a(w_i(t), w_i(t) - z) \leq 0 &\text{for all } z \in K_i(t), \text{ a.e. } t \in [0, T].
\end{cases}
\end{aligned}
\]  

(3.3)
Hence, for 

\[ \begin{align*}
&\text{as is easily seen, the functions } w_i(t) = w_0, \quad w_i(t) = 0 \quad \text{resp.} \quad \frac{\partial w_i(t)}{\partial n} = 0 \quad \text{on } \Gamma \text{ for a.e. } t \in [0, T].
\end{align*} \]

Now, for any fixed 0 < s ≤ T put 

\[ C(s) := \max \{ |w_{01} - w_{02}|_{C(\Omega)}, |\psi_1^1 - \psi_1^2|_{C(\Omega)}, |\psi_2^1 - \psi_2^2|_{C(\Omega)} \}. \]

As is easily seen, the functions \( w_i(t) - (w_1(t) - w_2(t) - C(s))^+ \) and \((w_1(t) - w_2(t) - C(s))^+ + w_2(t) \) are able to be taken as z in (3.3) for \( i = 1, 2 \), respectively. By adding two inequalities obtained from (3.3) with these test functions as z we have 

\[ \frac{1}{2} \frac{d}{dt} |(w_1(t) - w_2(t) - C(s))^+|_{L^2(\Omega)}^2 \leq 0 \quad \text{for a.e. } t \in [0, s]. \]

Since \((w_1(t) - w_2(t) - C(s))^+ = 0 \) a.e. on \( \Omega \), it follows that \( |(w_1(t) - w_2(t) - C(s))^+|_{L^2(\Omega)} = 0 \), hence \( w_1(t) - w_2(t) \leq C(s) \) everywhere on \( \Omega \) for all \( t \in [0, s] \). Similarly, \( w_2(t) - w_1(t) \leq C(s) \) everywhere on \( \Omega \) for all \( t \in [0, s] \). Thus 

\[ |w_1(t) - w_2(t)|_{C(\Omega)} \leq \max \{ |w_{01} - w_{02}|_{C(\Omega)}, |\psi_1^1 - \psi_1^2|_{C(\Omega)}, |\psi_2^1 - \psi_2^2|_{C(\Omega)} \} \quad \text{(3.4)} \]

for all \( 0 \leq t \leq s \leq T \). For a detail proof of (3.4), see [6]. Note here that 

\[ |\psi_i^1 - \psi_i^2|_{C(\Omega)} \leq L_1 |u_1 - u_2|_{C(\Omega)} \quad \text{for } i = 1, 2, \]

where \( L_1 := \max \{ \text{Lip}(f_a), \text{Lip}(f_d) \} \) with Lipschitz constant \( \text{Lip}(f_j) \) of \( f_j \), \( j = a, d \). Combining (3.4) with this inequality we have (3.2). \( \square \)

By using Lemma 3.1 we prove Theorem 2.1.

**Proof of Theorem 2.1.** Let \( T_0 \) be any number with 0 < \( T_0 < T \). Then, by Lemma 3.1, for all \( t \in [0, T_0] \)

\[ |u_1(t) - u_2(t)|_{C(\Omega)} \leq |u_{01} - u_{02}|_{C(\Omega)} + T_0 |w_1 - w_2|_{C(\Omega)}, \]

which shows that for all \( t \in [0, T_0] \)

\[ (1 - T_0L_1) |u_1(t) - u_2(t)|_{C(\Omega)} \leq (1 - T_0L_1) |u_{01} - u_{02}|_{C(\Omega)} \]

\[ \leq |u_{01} - u_{02}|_{C(\Omega)} + T_0 |w_0|_{C(\Omega)} - w_{02}|_{C(\Omega)}. \]

Hence, for \( T_0 = \frac{1}{2T} \),

\[ |u_1(t) - u_2(t)|_{C(\Omega)} \leq 2(1 + T_0) |u_{01} - u_{02}|_{C(\Omega)} + |w_0|_{C(\Omega)} - w_{02}|_{C(\Omega)}. \]
for all \( t \in [0, T_0] \). By the way, we have
\[
|w_1(t) - w_2(t)|_{C(\overline{\Omega})} \\
\leq |w_0 - w_02|_{C(\overline{\Omega})} + L_1|u_1 - u_2|_{C(\overline{\Omega})} \\
\leq |w_0 - w_02|_{C(\overline{\Omega})} + 2L_1(1 + T_0)(\|u_01 - u_02\|_{C(\overline{\Omega})} + |w_01 - w_02|_{C(\overline{\Omega})})
\]
for all \( t \in [0, T_0] \). Therefore, with \( L_2 := 2(1 + L_1)(1 + T_0) + 1 \),
\[
|u_1(t) - u_2(t)|_{C(\overline{\Omega})} + |w_1(t) - w_2(t)|_{C(\overline{\Omega})} \\
\leq L_2(|u_01 - u_02|_{C(\overline{\Omega})} + |w_01 - w_02|_{C(\overline{\Omega})})
\]
for all \( t \in [0, T_0] \).

By the same argument as above,
\[
|u_1(t) - u_2(t)|_{C(\overline{\Omega})} + |w_1(t) - w_2(t)|_{C(\overline{\Omega})} \\
\leq L_2(|u_1(T_0) - u_2(T_0)|_{C(\overline{\Omega})} + |w_1(T_0) - w_2(T_0)|_{C(\overline{\Omega})}) \\
\leq L_2^2(|u_01 - u_02|_{C(\overline{\Omega})} + |w_01 - w_02|_{C(\overline{\Omega})})
\]
for all \( t \in [T_0, 2T_0] \). Repeating this procedure finitely many times, we get
\[
|u_1(t) - u_2(t)|_{C(\overline{\Omega})} + |w_1(t) - w_2(t)|_{C(\overline{\Omega})} \\
\leq L_2^n(|u_01 - u_02|_{C(\overline{\Omega})} + |w_01 - w_02|_{C(\overline{\Omega})})
\]
for all \( t \in [0, T] \), where \( n_0 \) is a positive integer with \( n_0 T_0 \geq T \). It is easy to see that (2.4) holds for \( A(T) := L_2^n \). \( \square \)

4. Approximate problems

In this section we consider regular approximation for our original problems. A solution of \( P_{\omega}(u_0, w_0) \) (resp. \( P_{\omega^u}(u_0, w_0) \)) will be constructed as a limit of solutions \( \{u_{\lambda}, w_{\lambda}\} \) of approximate problems, referred as \( P_{\omega_{\lambda}}(u_0, w_0) \) (resp. \( P_{\omega^u_{\lambda}}(u_0, w_0) \)) with approximating parameter \( 0 < \lambda \leq 1 \), of the following form:
\[
u_{\lambda}' - \Delta u_{\lambda}(t) + w_{\lambda} = 0 \quad \text{in} \quad L^2(\Omega), \quad t > 0, \quad (4.1)
\]
\[
u_{\lambda}' - \kappa \Delta w_{\lambda}(t) + \hat{\omega}(I_{\lambda}(t); u_{\lambda}(t)) = 0 \quad \text{in} \quad L^2(\Omega), \quad t > 0, \quad (4.2)
\]
\[
\nu_{\lambda}(t) = 0, \quad w_{\lambda}(t) = 0 \quad \text{(resp.} \quad \hat{\nu}(\nu_{\lambda}) = 0, \quad \hat{w}(\nu_{\lambda}) = 0 \text{)} \quad \text{on} \Gamma, \quad t > 0, \quad (4.3)
\]
\[
u_{\lambda}(0) = u_0, \quad w_{\lambda}(0) = w_0 \quad \text{in} \quad L^2(\Omega), \quad (4.4)
\]
where \( \hat{\omega}(I_{\lambda};(w_{\lambda})) \) is a Lipschitz continuous mapping from \( L^2(\Omega) \times L^2(\Omega) \) into \( L^2(\Omega) \) with respect to \( u_{\lambda}, w_{\lambda} \), defined by the following formula
\[
\hat{\omega}(I_{\lambda};z) = \frac{(z - f_0(v)) +}{\lambda} - \frac{(f_0(v) - z) +}{\lambda}, \quad v, z \in L^2(\Omega);
\]
this is the so-called Yosida-regularization of the subdifferential \( \partial I_v \) of the convex function \( I_v \) on \( L^2(\Omega) \), and it is the subdifferential of the convex function \( (I_v)_\lambda(z) \) of \( C^1 \) class defined by
\[
(I_v)_\lambda(z) = \frac{1}{2\lambda} \left( (z - f_d(v))^+ |_{L^2(\Omega)} + |(f_a(v) - z)^+ |_{L^2(\Omega)} \right).
\]

We assume in this section that the initial data \( u_0 \) and \( w_0 \) are good enough, for instance they satisfy that
\[
u_0, \omega_0 \in C^2(\bar{\Omega}) \quad \text{with} \quad u_0 = w_0 = 0 \quad \text{and} \quad \frac{\partial u_0}{\partial n} = \frac{\partial w_0}{\partial n} = 0 \quad \text{on} \quad \Gamma. \quad (4.5)
\]

By the semilinear theory of PDEs (cf. [7–9]), for each \( 0 < \lambda \leq 1 \) and \( 0 < T < +\infty \), problem \( P_{\lambda, \Omega}^T(u_0, w_0) \) (resp. \( P_{\lambda, N}^T(u_0, w_0) \)) has one and only one classical solution \( \{u_\lambda, w_\lambda\} \) such that
\[
u_\lambda \in C^2(\bar{Q}_T), \quad w_\lambda \in C([0,T]; C^2(\bar{\Omega})), \quad w'_\lambda \in C(\bar{Q}_T).
\]

Throughout this section, in addition to (2.1) we assume that
\[
f_a \quad \text{and} \quad f_d \quad \text{are bounded on} \quad \mathbb{R}. \quad (4.6)
\]

Under conditions (2.1), (4.5) and (4.6) we now give some uniform estimates for approximate solutions \( \{u_\lambda, w_\lambda\} \) with respect to parameter \( \lambda \) and various norms of initial data \( u_0, w_0 \).

Lemma 4.1. (i) \( |w_\lambda|_{C(\overline{Q}_T)} \leq \max\{\sup |f_a|, \sup |f_d|\} \) for all \( \lambda \in (0,1] \).

(ii) There is nonnegative nondecreasing function \( M_1: \mathbb{R}^3 \rightarrow \mathbb{R}_+ \) such that
\[
|u_\lambda|_{C(\overline{Q}_T)} + |u_\lambda|_{w^{1,\gamma}(0,T; L^2(\Omega))} + |u_\lambda|_{C([0,T]; H^1(\Omega))} \leq M_1(T, |u_0|_{H^1(\Omega)}, |w_0|_{C(\overline{\Omega})}) \quad (4.7)
\]
for all \( 0 < \lambda \leq 1 \).

Proof. Put
\[
c_0 := \max\{\sup |f_a|, \sup |f_d|\},
\]
which is finite by (4.6). Then, multiply (4.2) by \( (w_\lambda(t) - c_0)^+ \) to have
\[
\frac{1}{2} \frac{d}{dt} \left| (w_\lambda(t) - c_0)^+ \right|^2_{L^2(\Omega)} + \kappa |\nabla (w_\lambda(t) - c_0)^+|^2_{L^2(\Omega)}
\]
\[
+ \left( \frac{(w_\lambda(t) - f_d(u_\lambda(t)))^+}{\lambda}, (w_\lambda(t) - c_0)^+ \right) \leq 0.
\]

Since the second and third terms are nonnegative and \( w_0 \leq c_0 \) on \( \Omega \), by integrating the above inequality in \( t \) over \( [0,s] \) with \( 0 \leq s \leq T \) we get
\[
|(w_\lambda(s) - c_0)^+|_{L^2(\Omega)} = 0 \quad \text{for all} \quad s \in [0,T], \quad \text{i.e.} \quad w_\lambda \leq c_0 \quad \text{on} \quad Q_T.
Similarly, we have \( w_j \geq -c_0 \) on \( Q_T \). Thus
\[
|w_j|_{C(Q_T)} \leq c_0.
\]  
(4.8)

On account of (4.8), by a similar argument for equation (4.1) as above we have
\[
|u_j|_{C(Q_T)} \leq c_0 T + |u_0|_{C(\bar{\Omega})} =: c_1.
\]

Also, it follows from the usual energy estimates for (4.1) that
\[
|u_j|_{W^{2,2}(0, T; L^2(\Omega)))} + |u_j|_{C([0, T], H^1(\Omega))} \leq M'_j := M'_1(c_0, T, |u_0|_{H^1(\Omega)}),
\]
where \( M'_j : \mathbb{R} \to \mathbb{R}_+ \) is a nonnegative nondecreasing function. Thus (4.7) holds for
\[
M_1 := M'_1 + c_1.
\]

Next, for each \( 0 < \lambda \leq 1 \) and \( 0 \leq t \leq T \) we define
\[
X_j(t) := \frac{1}{2} |u_j(t)|^2_{L^2(\Omega)} + \frac{\kappa}{2} |\nabla w_j(t)|^2_{L^2(\Omega)} + 2(I_{u_j(\cdot)})_j(w_j(t)).
\]  
(4.9)

**Lemma 4.2.** There are constants \( \delta_1 > 0 \) and \( R_1 > 0 \), independent of initial data \( u_0, w_0, \kappa \in (0, \kappa_0) \) and \( \lambda \in (0, 1) \), such that
\[
\delta_1 |u_j'(t)|^2_{L^2(\Omega)} + \delta_1 |\hat{c}(I_{u_j(\cdot)})_j(w_j(t))|^2_{L^2(\Omega)} + |\nabla u_j'(t)|^2_{L^2(\Omega)} + \frac{d}{dt} X_j(t)
\]
\[
\leq R_1 |u_j'(t)|^2_{L^2(\Omega)} + R_1 \kappa |u_j'(t)|^4_{L^2(\Omega)} + |w_j(t)|^{4}_{L^2(\Omega)} + 1 \quad \text{for all } t \in [0, T].
\]  
(4.10)

**Proof.** For simplicity we write \( u, w \) for \( u_j, w_j \). Let us compute
\[
(4.2) \times w', \quad (4.2) \times \hat{c}(I_{u_j})_j(w), \quad \frac{d}{dt}(4.1) \times u'.
\]

First, \((4.2) \times w'\) yields that
\[
|w'(t)|^2_{L^2(\Omega)} + \frac{\kappa}{2} |\nabla w(t)|^2_{L^2(\Omega)} + \frac{d}{dt} (I_{u_j})_j(w(t))
\]
\[
\leq L_1 |u_j'(t)|_{L^2(\Omega)} |\hat{c}(I_{u_j(\cdot)})_j(w(t))|_{L^2(\Omega)},
\]  
(4.11)

where \( L_1 := \max\{\text{Lip}(f_a), \text{Lip}(f_d)\} \).

Secondly, by computing \((4.2) \times \hat{c}(I_{u_j})_j(w)\) we have
\[
\frac{d}{dt} (I_{u_j})_j(w(t)) + \kappa \hat{c}(I_{u_j(\cdot)})_j(w(t)) \leq L_1 |u_j'(t)|_{L^2(\Omega)} |\hat{c}(I_{u_j(\cdot)})_j(w(t))|_{L^2(\Omega)} + \kappa \left( \Delta f_d(u(t)), \frac{(w(t) - f_d(u(t)))^+}{\lambda} \right)
\]
\[
- \kappa \left( \Delta f_d(u(t)), \frac{(f_d(u(t)) - w(t))^+}{\lambda} \right).
\]  
(4.12)
Thirdly, \( \frac{d}{dt}(4.1) \times u'' \) gives

\[
\frac{1}{2} \frac{d}{dt} |u'(t)|_{L^2(\Omega)}^2 + |\nabla u'(t)|_{L^2(\Omega)}^2 \leq |w'(t)|_{L^2(\Omega)} |u'(t)|_{L^2(\Omega)}. \tag{4.13}
\]

Now, let \( \sigma \) be any positive number. Then, using the Young’s inequality \( ab \leq \sigma a^2 + \frac{1}{4\sigma} b^2 \) for any nonnegative numbers \( a, b \), we see from the sum (4.11) + (4.12) + (4.13) that

\[
(1 - \sigma) |w'(t)|_{L^2(\Omega)}^2 + (1 - 2\sigma - 2\kappa \lambda \lambda) \varphi(I_{u(t)}), (w(t))_{L^2(\Omega)}^2 + |\nabla u'(t)|_{L^2(\Omega)}^2 + \frac{d}{dt} X_{f}(t)
\leq \left( \frac{L_2^2}{2\sigma} + \frac{1}{4\sigma} \right) |u'(t)|_{L^2(\Omega)}^2 + \frac{\kappa}{4\sigma} (|\Delta f_\delta(u(t))|_{L^2(\Omega)}^2 + |\Delta f_{\delta}(u(t))|_{L^2(\Omega)}^2). \tag{4.14}
\]

Here \( |\Delta f_\delta(u(t))|_{L^2(\Omega)} \) and \( |\Delta f_{\delta}(u(t))|_{L^2(\Omega)} \) are estimated by \( |u'|_{L^2(\Omega)} \) as follows. Note that

\[
\Delta f_\delta(u) = f''_\delta(u) |\nabla u|^2 + f'_\delta(u) \Delta u \quad \text{a.e. on } Q_T,
\]

so that

\[
|\Delta f_\delta(u(t))|_{L^2(\Omega)}^2 \leq 2L_2^2 (|\nabla u(t)|_{L^2(\Omega)}^2 + |\Delta u(t)|_{L^2(\Omega)}^2) \tag{4.15}
\]

where \( L_2 := \max \{ L_1, |f''_\delta|_{L^\infty(\mathbb{R})}, |f'_\delta|_{L^\infty(\mathbb{R})} \} \). Since \( u' = \Delta u - w \), we have

\[
|\varphi_0 w(t)|_{L^2(\Omega)}^2 \leq L_3 (|u'(t)|_{L^2(\Omega)}^2 + |w(t)|_{L^2(\Omega)}^2), \tag{4.16}
\]

where \( \varphi_0 \) is the projection from \( L^2(\Omega) \) onto the closed subspace \( \{ z \in L^2(\Omega); \int_{\Omega} z \, dx = 0 \} \) and \( L_3 \) is a positive constant independent of initial data \( u_0, w_0, \kappa \in (0, \kappa_0] \) and \( \lambda \in (0, 1) \). Also, by (4.16) and the Sobolev embedding theorem,

\[
|\nabla u(t)|_{L^4(\Omega)}^4 \leq L_4 |\varphi_0 w(t)|_{L^2(\Omega)}^4 \leq L_5 (|u'(t)|_{L^2(\Omega)}^4 + |w(t)|_{L^2(\Omega)}^4), \tag{4.17}
\]

where \( L_4, L_5 \) are positive constants having the same properties as \( L_3 \). Combining (4.15) and (4.17), we obtain that

\[
|\Delta f_\delta(u(t))|_{L^2(\Omega)}^2 \leq L_6 (|u'(t)|_{L^2(\Omega)}^4 + |w(t)|_{L^2(\Omega)}^4 + 1); \tag{4.18}
\]

similarly,

\[
|\Delta f_{\delta}(u(t))|_{L^2(\Omega)}^2 \leq L_6 (|u'(t)|_{L^2(\Omega)}^4 + |w(t)|_{L^2(\Omega)}^4 + 1), \tag{4.19}
\]

where \( L_6 \) is a positive constant having the same properties as \( L_3 \). Therefore it follows from (4.14), (4.18) and (4.19) that (4.10) holds for

\[
\delta_1 := 1 - 2\sigma - 2\sigma \kappa_0, \quad R_1 := \max \left\{ \frac{L_2^2}{2\sigma} + \frac{1}{4\sigma} \frac{L_6}{2\sigma} \right\}
\]

with a sufficiently small \( \sigma > 0 \). \( \square \)
Corollary 4.1. Let $X_\lambda$ be the function defined by (4.9). Then, there is a positive constant $R_2$, independent of initial data $u_0, w_0, \kappa \in (0, \kappa_0]$ and $\lambda \in (0, 1]$, such that

$$\frac{d}{dt} X_\lambda(t) \leq R_2 (|u'_\lambda(t)|^2_{L^2(\Omega)} + 1) X_\lambda(t) + R_2 (|w_2(t)|^2_{L^2(\Omega)} + 1) \text{ for all } t \in [0, T];$$

(4.20)

in fact, we can take $R_2 := 2(\kappa_0 + 1)R_1$.

Inequality (4.20) is an immediate consequence of (4.10).

Lemma 4.3. There is a nonnegative nondecreasing function $M_2: \mathbb{R}^3 \to \mathbb{R}_+$ such that

$$|u_\lambda|_{C([0,T];L^2(\Omega))} + |u'_\lambda|_{L^2(0,T;H^1(\Omega))} + |w_2|_{C([0,T];H^1(\Omega))} + |w'_2|_{L^2(0,T;L^2(\Omega))} + |\tilde{c}(u_\lambda) x(w_2)|_{L^2(0,T;L^2(\Omega))} \leq M_2(T, |u_0|_{H^2(\Omega)}, |w_0|_{H^1(\Omega)}) \text{ for all } \lambda \in (0, 1].$$

(4.21)

Proof. It follows from (4.20) that

$$X_\lambda(t) \leq X_\lambda(0) \exp \left\{ \int_0^T R_2 (|u'_\lambda|^2_{L^2(\Omega)} + 1) \, ds \right\} + R_2 \frac{1}{2} \int_0^T T (|w_2|^4_{L^2(\Omega)} + 1) \, ds \text{ for all } t \in [0, T].$$

(4.22)

We note that

$$X_\lambda(0) := \frac{1}{2} |\Delta u_0 - w_0|^2_{L^2(\Omega)} + \frac{\kappa}{2} |\nabla w_0|^2_{L^2(\Omega)},$$

and by Lemma 4.1, $\{w_2\}$ and $\{u'_\lambda\}$ are bounded in $C([0, T]; L^2(\Omega))$ and $L^2(0, T; L^2(\Omega))$, respectively. Therefore, (4.22) shows that $\{X_\lambda\}$ is uniformly bounded on $[0, T]$, so that $\{u'_\lambda\}$ and $\{w_2\}$ are, respectively, bounded in $C([0, T]; L^2(\Omega))$ and $C([0, T]; H^1(\Omega))$, and (4.21) holds for a suitable function $M_2(T, |u_0|_{H^2(\Omega)}, |w_0|_{H^1(\Omega)}).$ \Box

Lemma 4.4. There is a nonnegative nondecreasing function $M_3: \mathbb{R}^3 \to \mathbb{R}_+$ such that

$$|u'_\lambda|_{C([0,T];L^2(\Omega))} + |u_{\lambda}|_{C([0,T];H^1(\Omega))} + |w'_2|_{C([0,T];L^2(\Omega))} + |w_2|_{C([0,T];H^1(\Omega))} \leq M_3(T, |u_0|_{C^2(\overline{\Omega})}, |w_0|_{C^1(\overline{\Omega})}) \text{ for all } \lambda \in (0, 1].$$

(4.23)

Proof. Applying the same technique as in the proofs of Lemma 4.1 and Theorem 2.1 to the approximate problem (4.1)–(4.4), we can show for each $\lambda > 0$ that

$$|u_\lambda(t + \Delta t) - u_\lambda(t)|_{C^2(\overline{\Omega})} + |w_2(t + \Delta t) - w_2(t)|_{C^1(\overline{\Omega})} \leq A(T) (|u_\lambda(\Delta t) - u_\lambda(\Delta t)|_{C^2(\overline{\Omega})} + |w_2(\Delta t) - w_2(\Delta t)|_{C^1(\overline{\Omega})}) \text{ for all } t \in [0, T]$$

and $\Delta t > 0$ with $t + \Delta t \in [0, T].$
where $A(T)$ is the same constant as in Theorem 2.1. Since $u_j$ and $w_j$ are of $C^1$ on $\overline{Q_T}$, it follows from (4.24) that
\[
|u_j'(t)|_{C(\overline{\Omega})} + |w_j'(t)|_{C(\overline{\Omega})} \leq A(T)\{|u_j'(0)|_{C(\overline{\Omega})} + |w_j'(0)|_{C(\overline{\Omega})}\}
\]
\[
= A(T)\{\Delta u_0 - w_0|_{C(\overline{\Omega})} + |\Delta w_0|_{C(\overline{\Omega})}\}
\]
\[
\leq A(T)\{|u_0|_{C(\overline{\Omega})} + |w_0|_{C(\overline{\Omega})}\}
\]
for all $t \in [0, T]$, so that
\[
|u_j|_{C([0,T];C(\overline{\Omega}))}, \quad |w_j|_{C([0,T];C(\overline{\Omega}))} \leq A(T)\{|u_0|_{C(\overline{\Omega})} + |w_0|_{C(\overline{\Omega})}\}. \tag{4.25}
\]
Since $-\Delta u_j(t) = -u_j'(t) - w_j(t)$, it follows that
\[
|u_j(t)|_{H^2(\Omega)} \leq R_3(|u_j'(t)|_{L^2(\Omega)} + |w_j(t)|_{L^2(\Omega)}), \tag{4.26}
\]
where $R_3$ is a positive constant independent of $\lambda \in (0,1]$, $t \in [0, T]$ and initial data. Therefore, by (4.25) and estimates of Lemma 4.1,
\[
|u_j|_{C([0,T];H^2(\Omega))} \leq R_4(|u_0|_{C(\overline{\Omega})} + |w_0|_{C(\overline{\Omega})} + 1), \tag{4.27}
\]
where $R_4(T)$ is a positive constant depending only on $T$.

Next, we estimate $|w_j(t)|_{H^2(\Omega)}$. Multiplying equation (4.2) by $\tilde{c}(I_{u_j},\lambda)(w_j)$, we get
\[
(w_j'(t), (\tilde{c}I_{u_j})(w_j(t)) + \kappa \lambda \nabla \tilde{c}(I_{u_j},\lambda)(w_j(t))|_{L^2(\Omega)}
\]
\[
-\kappa \Delta f_0(u_j(t)), (w_j(t) - f_0(u_j(t)))^+ \right) + \kappa \left( \Delta f_0(u_j(t)), \frac{(f_0(u_j(t)) - w_j(t))^+}{\lambda} \right)
\]
\[
+ |\tilde{c}(I_{u_j},\lambda)(w_j(t))|^2_{L^2(\Omega)} = 0.
\]
From this we obtain that
\[
|\tilde{c}(I_{u_j},\lambda)(w_j(t))|_{L^2(\Omega)} \leq |w_j'(t)|_{L^2(\Omega)} + \kappa \lambda \Delta f_0(u_j(t))|_{L^2(\Omega)} + \kappa \lambda \Delta f_0(u_j(t))|_{L^2(\Omega)}
\]
for all $t \in [0, T]$. On account of (4.25), (4.26) with (4.18), (4.19) the right hand side of the above inequality is dominated by $R_5(T)(|u_0|_{C(\overline{\Omega})}^2 + |w_0|_{C(\overline{\Omega})}^2 + 1)$, where $R_5(T)$ is a positive constant depending only on $T$. This estimate with (4.25) implies that
\[
|w_j|_{C([0,T];H^2(\Omega))} \leq R_6(|u_0|_{C(\overline{\Omega})}^2 + |w_0|_{C(\overline{\Omega})}^2 + 1), \tag{4.28}
\]
where $R_6(T)$ is a positive constant depending only on $T$.

Estimates (4.25), (4.27) and (4.28) imply that (4.23) holds with
\[
M_3 := (2A(T) + R_4(T) + R_6(T))(|u_0|_{C(\overline{\Omega})}^2 + |w_0|_{C(\overline{\Omega})}^2 + 2).
\]

5. Convergence of approximate solutions and existence of a solution

We show the existence of a strong solution of $P_{P\lambda}(u_0, w_0)$ (resp. $P_{P\lambda}(u_0, w_0)$) in three steps. In the first step, under additional assumptions (4.5) and (4.6), we show the
convergence of approximate solutions \( \{u_n, w_n\} \), which was constructed in the previous section, as \( \lambda \) tends to 0, and show that the limit is a strong solution of our problem.

In the second step, we show the existence of a strong solution of our problem under conditions (2.3), (2.7) and (4.6). In the final step, we show it without (4.6).

Step 1. Assume that (4.5) and (4.6) hold. Then, by virtue of Lemmas 4.1, 4.3 and 4.4 there exists a sequence \( \{u_n := u_{\lambda_n}, w_n := w_{\lambda_n}\} \) and \( \{\xi_n := \partial I_{u_n}(w_n)\} \) converge to some couple of functions \( \{u, w\} \) and a function \( \xi \), respectively, in the sense that

\[
\begin{align*}
  u_n &\to u \quad w_n \to w \quad \text{in} \quad C(\overline{\Omega_T}), \\
  u_n &\rightharpoonup u \quad \text{weakly in} \quad W^{1,2}(0, T; H^1(\Omega)), \\
  w_n &\rightharpoonup w \quad \text{weakly in} \quad W^{1,2}(0, T; H^2(\Omega)), \\
  \xi_n &\to \xi \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)).
\end{align*}
\]

Since \( w_n' - \kappa \Delta w_n = -\xi_n \) and \( u_n' - \Delta u_n = -w_n \), convergences (5.1), (5.3) and (5.4) show by the well-posedness for linear heat equations that

\[
\begin{align*}
  w_n &\to w \quad \text{in} \quad L^2(0, T; H^1(\Omega)), \\
  u_n' &\to u' \quad \text{in} \quad C([0, T]; H^2(\Omega)), \\
  u_n &\to u \quad \text{in} \quad C([0, T]; H^1(\Omega)),
\end{align*}
\]

We now show that the limit \( \{u, w\} \) obtained above is a solution of \( P_{\varepsilon_D}(u_0, w_0) \) (resp. \( P_{\varepsilon_N}(u_0, w_0) \)).

Passing to the limit as \( \lambda = \lambda_n \) in (4.1)–(4.4), we see by convergences (5.1)–(5.6) that the limit function \( \{u, w\} \) satisfies all the required properties in Definition 2.1, except

\[
\xi(t) \in \partial I_{\varepsilon_D}(w(t)) \quad \text{for a.e.} \ t \in [0, T];
\]

the boundary conditions \( u = w = 0 \) (resp. \( \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 \)) on \( \Sigma_T \) are satisfied in the sense of traces, since \( u_n = w_n = 0 \) (resp. \( \frac{\partial u_n}{\partial n} = \frac{\partial w_n}{\partial n} \)) on \( \Sigma_T \) and \( u_n \to u, \ w_n \to w \) in \( L^2(0, T; H^1(\Omega)) \) (resp. weakly in \( L^2(0, T; H^2(\Omega)) \)). To accomplish the existence proof of a strong solution, it remains to show that for a.e. \( t \in [0, T] \)

\[
\begin{align*}
  f_\varepsilon(u(t)) \leq w(t) &\leq f_\varepsilon(u(t)) \quad \text{on} \quad \Omega, \\
  (\xi(t), z - w(t)) &\leq 0 \quad \text{for all} \ z \in L^2(\Omega) \quad \text{with} \quad f_\varepsilon(u(t)) \leq z \leq f_\varepsilon(u(t)) \quad \text{a.e. on} \ \Omega,
\end{align*}
\]

which is equivalent to (5.7). This is proved by (5.1), (5.3) and (5.4) as follows. First, since

\[
(w_n - f_\varepsilon(u_n))^+ - (f_\varepsilon(u_n) - w_n)^+ = \lambda_n \xi_n \to 0 \quad \text{in} \quad L^2(0, T; L^2(\Omega)),
\]

\[
\begin{align*}
  w_n &\rightharpoonup 0 \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)), \\
  w_n &\rightharpoonup 0 \quad \text{weakly} \quad \text{in} \quad L^1(0, T; L^2(\Omega)).
\end{align*}
\]
it follows that \((w - f_d(u))^+ - (f_a(u) - w)^+ = 0\) on \(Q_T\), i.e. \(f_d(u(t)) \leq w(t) \leq f_a(u(t))\) on \(\Omega\) for all \(t \in [0, T]\). Next, let \(v\) be any function in \(L^2(0, T; L^2(\Omega))\) such that \(f_d(u) \leq v \leq f_a(u)\) a.e. on \(Q_T\). For each \(n\) put

\[ v_n := \max \{ \min \{ f_d(u_n), v \}, f_a(u_n) \} . \]

Then it is easy to see that \(f_a(u_n) \leq v_n \leq f_d(u_n)\) a.e. on \(Q_T\), and \(v_n \to v\) in \(L^2(0, T; L^2(\Omega))\) by \((5.1)\). We observe that

\[ \int_0^T (\zeta_n, v_n - w_n) \, dt \leq \int_0^T (I_{u_n} \zeta_n (v_n) - (I_{u_n} \zeta_n (w_n)) \, dt \leq - \int_0^T (I_{u_n} \zeta_n (w_n) \, dt \leq 0, \]

since \((I_{u_n(t)} \zeta_n (v_n(t)) = 0\). Letting \(n \to +\infty\) in the above inequality yields that

\[ \int_0^T (\zeta, v - w) \, dt \leq 0. \]

This implies by the arbitrariness of \(v\) that \((5.8)\) (hence \((5.7)\)) holds for a.e. \(t \in [0, T]\).

By the way, by taking \(w\) as \(v\) in the above argument we see that

\[ \lim_{n \to +\infty} \int_0^T (I_{u_n} \zeta_n (w_n) \, dt = 0. \quad (5.9) \]

**Step 2.** Assume that \((2.3)\), \((2.7)\) and \((4.6)\) hold. In this case, choose a sequence \(\{u_{0n}, w_{0n}\}\) of good initial data satisfying properties in \((4.5)\) such that

\[ u_{0n} \to u_0 \quad \text{in} \quad H^2(\Omega), \quad w_{0n} \to w_0 \quad \text{in} \quad H^1(\Omega) \cap C(\overline{\Omega}) \quad \text{as} \quad n \to +\infty. \]

For each \(n\), by Step 1, there is a unique solution \(\{u_n, w_n\}\) of \(P_{kD}(u_{0n}, w_{0n})\) (resp. \(P_{kN}(u_{0n}, w_{0n})\)); for each \(n\) we denote by \(\zeta_n\) the function \(\zeta\) in \((v3)\) of Definition 2.1.

From \((2.4)\) of Theorem 2.1 it follows that \(\{u_n\}\) and \(\{w_n\}\) are Cauchy sequences in \(C(Q_T)\). Now, denote by \(u\) and \(w\) their limits, respectively. Then, just as in the first Step, for these limits \(u, w\) it is easy to see from Lemmas 4.1 and 4.3 that convergences \((5.1)-(5.6)\) hold and \(\{u, w\}\) is a strong solution of \(P_{kD}(u_0, w_0)\) (resp. \(P_{kN}(u_0, w_0)\)).

**Step 3.** In the case when \((4.6)\) is not necessarily satisfied, consider the following two (auxiliary) initial-boundary value problems

\[
\begin{align*}
\begin{cases}
  u'_j(t) - \Delta u_j(t) + f_j(u_j(t)) = 0 & \text{in} \ L^2(\Omega), \ a.e. \ t \in [0, T), \\
  u_j(t) = 0 \quad (\text{resp.} \quad \partial u_j(t) / \partial n = 0) & \text{on} \ \Gamma, \ a.e. \ t \in [0, T], \\
  u_j(0) = u_0 &
\end{cases}
\end{align*}
\]

for each \(j = a, d\). By the theory of semilinear parabolic PDEs (cf. [7]), this problem has one and only one (weak) solution \(u_j\) in \(L^\infty(Q_T) \cap C([0, T]; L^2(\Omega)) \cap C^2_{\text{loc}}(\overline{\Omega} \times (0, T))\) with \(u'_j \in C^2_{\text{loc}}(\overline{\Omega} \times (0, T))\), as long as \(u_0\) is prescribed in \(L^\infty(\Omega)\) for each \(j = a, d\). Now, choose a large constant \(N\) so that

\[ N > |u_a|_{L^\infty(Q_T)} + |u_d|_{L^\infty(Q_T)}. \]
and for this $N$ take two bounded functions $f_a^N$ and $f_d^N$ on $\mathbb{R}$ which satisfy the same properties in condition (2.1) and

$$f_a^N = f_a, \quad f_d^N = f_d \quad \text{on} \quad [-N,N].$$

We denote by $P_{a,d}(u_0,w_0)_N$ (resp. $P_{e,N}(u_0,w_0)_N$) the problem with $f_a$ and $f_d$ replaced by $f_a^N$ and $f_d^N$, respectively. Applying the result of Step 2 and Theorem 2.1, we see that this problem has a unique strong solution $\{u_N,w_N\}$. Next, compare $u_N$ with $u_a$ and $w_N$.

Then

$$u_d \leq u_N \leq u_a \quad \text{on} \quad QT. \quad \text{(5.11)}$$

In fact, multiplying $(u_N - u_a)' - \Delta (u_N - u_a) + (w_N - f_a(u_a)) = 0$ by $(u_N - u_a)^+$, we get

$$\frac{1}{2} \frac{d}{dt} \left| (u_N(t) - u_a(t))^+ \right|_{L^2(\Omega)}^2 + |\nabla (u_N(t) - u_a(t))^+|_{L^2(\Omega)}^2$$

$$+ (w_N(t) - f_a(u_a))^+ + (u_N(t) - u_a(t))^+ = 0.$$

Here, note that if $u_N > u_a$, then

$$w_N \geq f_a^N(u_N) \geq f_a^N(u_a) = f_a(u_a),$$

so that

$$(w_N - f_a(u_a))(w_N - u_a)^+ \geq 0 \quad \text{on} \quad QT.$$

This implies that $\frac{1}{2} \frac{d}{dt} \left| (u_N(t) - u_a) \right|_{L^2(\Omega)}^2 \leq 0$ on $(0,T]$, whence $|(u_N(t) - u_a) |_{L^2(\Omega)}^2 \leq 0$ for all $t \in [0,T]$. Accordingly, $u_N \leq u_a$ on $QT$. Similarly we have $u_d \leq u_N$ on $QT$. Thus (5.11) holds.

From (5.11) it follows that $f_a^N(u_a) = f_d^N(u_a)$ and $f_d^N(u_N) = f_d(u_N)$ on $QT$ and $\xi \in C^{1,A}(w_N)$, where $\xi$ is the function $\xi$ in condition (v3) of Definition 2.1 for $P_{a,d}(u_0,w_0)_N$ (resp. $P_{e,N}(u_0,w_0)_N$). Therefore $\{u_N,w_N\}$ is a strong solution of $P_{a,d}(u_0,w_0)$ (resp. $P_{e,N}(u_0,w_0)$), too. Thus the proof of existence of a strong solution has been accomplished.

Next we give some energy inequalities for strong solutions.

**Lemma 5.1.** Let $\kappa \in (0,\kappa_0)$ and assume that initial data $u_0, w_0$ satisfy (2.3) and (2.7). Let $\{u,w\}$ be the strong solution of $P_{\kappa,D}(u_0,w_0)$ (resp. $P_{\kappa,N}(u_0,w_0)$) on $[0,T]$ and $\xi$ be the functions as in (v3) of Definition 2.1. Then

$$\delta_1 \left| u'(t) \right|_{L^2(\Omega)}^2 + \delta_2 \left| \xi(t) \right|_{L^2(\Omega)}^2 + |\nabla u'(t)|_{L^2(\Omega)}^2 \leq \frac{1}{2} \frac{d}{dt} \left| u'(t) \right|_{L^2(\Omega)}^2 + \frac{\kappa}{2} |\nabla v(t) - u'(t)|_{L^2(\Omega)}^2$$

$$\leq R_1 \left| u'(t) \right|_{L^2(\Omega)}^2 + 2R_1 t \left| u'(t) \right|_{L^2(\Omega)}^2 + 2R_1 t + 1 \quad \text{for a.e.} \ t \in [0,T], \quad \text{(5.12)}$$
where \( \delta_1 \) and \( R_1 \) are the same positive constants as in Lemma 4.2, and

\[
\delta_1 \int_s^{s_1} |w'(t)|^2_{L^2(\Omega)} \, dt + \frac{d}{dt}\left\{ \frac{1}{2} t|u'(t)|^2_{L^2(\Omega)} + \frac{\kappa}{2} t|w(t)|^2_{L^2(\Omega)} \right\} \\
\leq R_1 (1 + \kappa |u'(t)|^2_{L^2(\Omega)} + t|u'(t)|^2_{L^2(\Omega)} + \kappa R_1 (|w(t)|^4_{L^2(\Omega)} + 1) \\
+ \frac{1}{2} |u'(t)|^2_{L^2(\Omega)} + \frac{\kappa}{2} |w(t)|^2_{L^2(\Omega)} \quad \text{for a.e. } t \in [0, T].
\]  

(5.13)

**Proof.** Let \( \{u_n, w_n\} \) be the same sequence of approximate solutions as in Step 2; convergences (5.1)–(5.6) hold as well. Besides, by taking a further subsequence of \( \{u_n, w_n\} \) if necessary, we may assume that

\[ w_n(t) \rightharpoonup w(t) \quad \text{in } H^1(\Omega) \quad \text{and} \quad (I_{u_n})_{n}(w_n(t)) \to 0 \quad \text{for a.e. } t \in [0, T]; \]

and the last convergence is due to (5.9). Now, take any \( s \in [0, T] \) with \( w_n(s) \to w(s) \) in \( H^1(\Omega) \) and \( (I_{u_n})_{n}(w_n(s)) \to 0 \), integrate (4.10) with \( \lambda = \hat{\lambda}_n \) over \([s, s_1]\) in time \( t \) for a.e. \( s_1 \in (s, T] \), and pass to the limit as \( n \to +\infty \). Then we get

\[
\delta_1 \int_s^{s_1} |w'|^2_{L^2(\Omega)} \, dt + \frac{d}{dt}\left\{ \frac{1}{2} t|u'(s)|^2_{L^2(\Omega)} + \frac{\kappa}{2} |w(s)|^2_{L^2(\Omega)} \right\} \\
\leq \frac{1}{2} |u'(s)|^2_{L^2(\Omega)} + \frac{\kappa}{2} |w(s)|^2_{L^2(\Omega)} + R_1 \int_s^{s_1} |u'|^2_{L^2(\Omega)} \, dx \\
+ \kappa R_1 \int_s^{s_1} (|u'|^4_{L^2(\Omega)} + |w|^4_{L^2(\Omega)} + 1) \, dt.
\]

Since \( w \in C([0, T]; H^1(\Omega)) \) and \( u' \in C([0, T]; L^2(\Omega)) \), the above inequality holds for every \( 0 \leq s \leq s_1 \leq T \). Hence it follows that the function \( t \to \frac{1}{2} |u'(t)|^2_{L^2(\Omega)} + \frac{\kappa}{2} |w(t)|^2_{L^2(\Omega)} \) is of bounded variation on \([0, T]\) and (5.12) holds a.e. on \([0, T]\). Also, (5.13) is immediately obtained by multiplying (5.12) by \( t \). \( \square \)

For our existence proof of a weak solution we prepare the following lemma.

**Lemma 5.2.** There is a nonnegative nondecreasing function \( M_4 : \mathbb{R}^2 \to \mathbb{R}_+ \), such that

\[
|w|_{C([0, T]; L^2(\Omega))} + \kappa^\frac{1}{2} |w|_{L^2(0, T; H^1(\Omega))} + |u|_{C([0, T]; H^1(\Omega))} + |u'|_{L^2(0, T; L^2(\Omega))} \leq M_4(T, |u_0|_{H^1(\Omega)})
\]  

(5.14)

for any strong solution \( \{u, w\} \) of \( P_{\varepsilon, \kappa}(u_0, w_0) \) (resp. \( P_{\varepsilon, \kappa}(u_0, w_0) \)) with \( \kappa \in (0, \kappa_0] \) and initial data \( \{u_0, w_0\} \) satisfying (2.3) and (2.7).
**Proof.** Multiplying the equation $u' - \Delta u + w = 0$ by $u$ and $u'$ and integrating them in time, we obtain that

$$
|u(s)|^2_{L^2(\Omega)} + 2 \int_0^s |\nabla u(t)|^2_{L^2(\Omega)} \, dt \leq |u_0|^2_{L^2(\Omega)} + 2 \int_0^s |w(t)|_{L^2(\Omega)} |u(t)|_{L^2(\Omega)} \, dt
$$

and

$$
\int_0^s |u'(t)|^2_{L^2(\Omega)} \, dt + |\nabla u(s)|^2_{L^2(\Omega)} \leq \int_0^s |u_0|^2_{L^2(\Omega)} + \int_0^s |w(t)|^2_{L^2(\Omega)} \, dt
$$

for all $s \in [0, T]$. Since $f_+(u) \leq w \leq f_-(u)$, by the Gronwall’s lemma these inequalities imply that

$$
|w|^2_{C([0,T];L^2(\Omega))} + |w|^2_{C([0,T];H^1(\Omega))} + |u'|^2_{L^2(0,T;L^2(\Omega))} \leq R_8(T)(|u_0|^2_{H^1(\Omega)} + 1),
$$

(5.15)

where $R_8(T)$ is a positive constant depending only on $T, f_+$ and $f_-$. Next, multiplying the equation $w - \kappa \Delta w + \xi = 0$ by $w - f_-(u)$, we have

$$
(w'(t), w(t) - f_-(u(t))) + \kappa(\nabla w(t), \nabla (w(t) - f_-(u(t)))) + (\xi(t), w(t) - f_-(u(t))) = 0
$$

for a.e. $t \in [0, T]$. Also, by definition, $(\xi, w - f_-(u)) \geq 0$. Hence

$$
\frac{d}{dt} \left\{ \frac{1}{2} |w(t)|^2_{L^2(\Omega)} - (w(t), f_-(u(t))) \right\} + \frac{\kappa}{2} |\nabla w(t)|^2_{L^2(\Omega)} \leq \frac{\kappa}{2} L_1^2 |\nabla u(t)|^2_{L^2(\Omega)} + L_1 |w(t)|_{L^2(\Omega)} |u'(t)|_{L^2(\Omega)}
$$

for a.e. $t \in [0, T]$. The integration of this inequality in time and estimates in (5.15) yield that

$$
\kappa |w|^2_{L^2(0,T;H^1(\Omega))} \leq R_8(T)(|u_0|^2_{H^1(\Omega)} + 1),
$$

(5.16)

where $R_8(T)$ is a positive constant depending only on $T, f_+$ and $f_-$. From (5.15) and (5.16) it is easy to obtain an inequality of the form (5.14) for a suitable nonnegative nondecreasing function $M_4 : \mathbb{R}^2 \to \mathbb{R}_+$. □

Finally, by making use of Lemmas 5.1 and 5.2 we complete the proof of Theorem 2.2.

**Proof of Theorem 2.2.** In order to finish the proof of Theorem 2.2 it remains to show the existence of a weak solution having property (2.6) under condition (2.3). Given initial data $u_0, w_0$, choose a sequence of initial data $\{u_{0n}, w_{0n}\}$ which satisfy properties (2.3) and (2.7) and such that

$$
u_0 \rightharpoonup u_0 \quad \text{in} \quad H^1(\Omega) \cap C(\overline{\Omega}), \quad w_0 \rightharpoonup w_0 \quad \text{in} \quad C(\overline{\Omega}).
$$

According to the results proved in Steps 1, 2, 3, for each $n$ problem $P_{x,D}(u_{0n}, w_{0n})$ (resp. $P_{x,N}(u_{0n}, w_{0n})$) has one and only one strong solution $\{u_n, w_n\}$; let $\xi_n$ be the function $\xi$ in (v3) of Definition 2.1.
First, by (2.4) of Theorem 2.1 we observe that \( \{u_n\} \) and \( \{w_n\} \) are Cauchy in \( H^1(\Omega) \), so that for some functions \( u, w \in C(\overline{\Omega_T}) \)

\[
u_n \to u, \quad w_n \to w \text{ in } C(\overline{\Omega_T}).
\]

Next, by Lemma 5.1,

\[
t\left\{ \delta_1|u'_n(t)|_{L^2(\Omega)}^2 + \delta_1|\xi_n(t)|_{L^2(\Omega)}^2 + |\nabla u'_n(t)|_{L^2(\Omega)}^2 \right\}
\]

\[
+ \frac{d}{dt} \left\{ \frac{1}{2}|u'_n(t)|_{L^2(\Omega)}^2 + \frac{\kappa}{2}|u_n(t)|_{L^2(\Omega)}^2 \right\}
\]

\[
\leq R_1(1 + \kappa|u'_n(t)|_{L^2(\Omega)}^2) \cdot t|u'_n(t)|_{L^2(\Omega)}^2 + \kappa R_1 t|u_n(t)|_{L^2(\Omega)}^4 + t + \frac{1}{2}|u'_n(t)|_{L^2(\Omega)}^2 + \frac{\kappa}{2}|\nabla w_n(t)|_{L^2(\Omega)}^2 \quad \text{for a.e. } t \in [0, T].
\]

(5.18)

Noting estimates in Lemma 5.2 and applying the Gronwall’s lemma to (5.18), we see that

\[
\{u_n\} \text{ is bounded in } C([0, T]; H^1(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)),
\]

\[
\{w_n\} \text{ is bounded in } L^2(0, T; H^1(\Omega)),
\]

\[
\{t^\frac{1}{2}u'_n\} \text{ is bounded in } C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
\]

\[
\{t^\frac{1}{2}w_n\} \text{ is bounded in } C([0, T]; H^1(\Omega)),
\]

and

\[
\{t^\frac{1}{2}u'_n\} \text{ and } \{t^\frac{1}{2} \xi_n\} \text{ are bounded in } L^2(0, T; L^2(\Omega)).
\]

Therefore, by the well-posedness of linear heat equations and the above estimates we can derive that

\[
u_n \rightharpoonup u \text{ weakly}^* \text{ in } L^\infty(0, T; H^1(\Omega))
\]

\[
\text{and weakly in } W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)),
\]

\[
w_n \rightharpoonup w \text{ weakly in } L^2(0, T; H^1(\Omega)),
\]

\[
t^\frac{1}{2}u'_n \rightharpoonup t^\frac{1}{2}u' \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; H^1(\Omega)),
\]

\[
t^\frac{1}{2}w_n \rightharpoonup t^\frac{1}{2}w \text{ weakly}^* \text{ in } L^\infty(0, T; H^1(\Omega)) \text{ and weakly in } L^2(0, T; H^2(\Omega)),
\]

\[
t^\frac{1}{2}w'_n \rightharpoonup t^\frac{1}{2}w' \text{ weakly in } L^2(0, T; L^2(\Omega)),
\]

and for some \( \xi \in L^2_{\text{loc}}(0, T; L^2(\Omega)) \), by taking a subsequence of \( \{ \xi_n \} \) if necessary,

\[
t^\frac{1}{2}\xi_n \rightharpoonup t^\frac{1}{2} \xi \text{ weakly in } L^2(0, T; L^2(\Omega)).
\]

(5.20)

Moreover, by a similar argument as in Step 1, it follows from convergences (5.17), (5.19) and (5.20) that the limit \( \{u, w\} \) with \( \xi \) fulfills conditions (v1)–(v3) of Definition 2.1 and (2.6) holds.
6. Asymptotic convergence in $\kappa$

In this section we prove Theorem 2.3.

Proof of Theorem 2.3. Let $\tilde{\xi}_n$ be the function $\xi$ as in (v3) of Definition 2.1 for $P_{kD}(u_0,w_0)$ (resp. $P_{kN}(u_0,w_0)$) and

$$Y_n(t) := \frac{1}{2}|u'_n(t)|^2_{L^2(\Omega)} + \frac{K}{2} |\nabla w_n(t)|^2_{L^2(\Omega)}, \quad t \in [0, T].$$

It follows from (5.12) in Lemma 5.1 that

$$\frac{d}{dt} Y_n(t) \leq R_2(|u'_n(t)|_{L^2(\Omega)}^2 + 1)Y_n(t) + R_2(|w_n(t)|_{L^2(\Omega)}^4 + 1) \quad \text{for a.e.} \ t \in [0, T],$$

where $R_2$ is the same constant as in Corollary 4.1. Applying Gronwall’s lemma to this inequality, we have

$$Y_n(s) \leq Y_n(0) \exp \left\{ R_2 \int_0^T (|u'_n|_{L^2(\Omega)}^2 + 1) \, dt \right\} + R_2T (|w_n|_{C([0,T];L^2(\Omega))}^4 + 1)$$

$$\times \exp \left\{ R_2 \int_0^T (|u'_n|_{L^2(\Omega)}^2 + 1) \, dt \right\} \quad \text{for all} \ s \in [0, T]. \quad (6.1)$$

Since $Y_n(0) = \frac{1}{2} |\Delta u_0 - w_0|_{L^2(\Omega)}^2 + \frac{\xi}{2} |\nabla w_0|_{L^2(\Omega)}^2$, we infer from (6.1) and estimates in Lemma 5.2 that $\kappa := \{ \kappa_n \}_{n \in (0, \kappa_0]}$ is uniformly bounded on $[0, T]$, so that $\{u'_n\}$ and $\{k \nabla w_k\}$ are bounded in $C^1([0,T];L^2(\Omega))$ and $C([0,T];L^2(\Omega))$, respectively. Furthermore, these facts imply by (5.12) in Lemma 5.1 again that $\{w'_n\}$, $\{\xi_n\}$ and $\{\nabla u'_n\}$ are bounded in $L^2(0,T;L^2(\Omega))$. Therefore, there are a null sequence $\{\kappa_n\}$ in $(0,\kappa_0)$ and a triple of functions $\{u_n,w_n,\xi_n\}$ such that

$$u_n := u_{\kappa_n} \rightharpoonup u \quad \text{weakly in} \quad W^{1,2}(0,T;H^1(\Omega)), \quad u'_n \rightharpoonup u' \quad \text{weakly* in} \quad L^\infty(0,T;L^2(\Omega)), \quad (6.2)$$

$$w_n := w_{\kappa_n} \rightharpoonup w \quad \text{weakly in} \quad W^{1,2}(0,T;L^2(\Omega)) \quad (6.3)$$

and

$$\xi_n := \xi_{\kappa_n} \rightharpoonup \xi \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \quad (6.4)$$

In particular, $\{\kappa_n \Delta w_n (\equiv w'_n + \xi_n)\}$ is bounded in $L^2(0,T;L^2(\Omega))$ and

$$\kappa_n \Delta w_n \rightharpoonup 0 \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)), \quad (6.5)$$

since

$$\left| \int_0^T (\kappa_n \Delta w_n, \eta) \, dt \right| \leq \kappa_n^{\frac{1}{2}} |\nabla w_n|_{L^2(0,T;L^2(\Omega))} \cdot \kappa_n^{\frac{3}{2}} |\nabla \eta|_{L^2(0,T;L^2(\Omega))} \rightarrow 0.$$
for every $\eta \in L^2(0,T;H^1_0(\Omega))$. Also, since $u'_n - \Delta u_n = -w_n$, it follows from (6.2) and (6.3) by the well-posedness for heat equations that

$$\begin{cases}
u_n(t) \rightarrow u(t) \text{ weakly in } H^2(\Omega) \text{ and uniformly in } t \in [0,T], \\
u'_n \rightarrow u' \text{ in } C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).
\end{cases}$$

(6.6)

It is straightforward to see from convergences (6.3)–(6.6) that

$$\begin{align*}
&\xi(t) \in \partial I_u(w(t)) \text{ for a.e. } t \in [0,T]. \\
&\text{This is proved as follows. Let } v \text{ be any function in } L^2(0,T;H^1_0(\Omega)) \text{ (resp. } L^2(0,T;H^1(\Omega))) \text{ such that } f_a(u) \leq v \leq f_d(u) \text{ a.e. on } \Omega, \text{ and put for each } n \\
&v_n := \max\{\min\{v,f_d(u_n)\},f_a(u_n)\}.
\end{align*}$$

Clearly, $f_a(u_n) \leq v_n \leq f_d(u_n)$ a.e. on $\Omega$ and $v_n \rightarrow v$ in $L^2(0,T;L^2(\Omega))$. Therefore, since $\xi_n \in \partial I_{u_n}(w_n)$, we observe that

$$\int_0^T (\xi_n, v_n - w_n) \, dt \leq 0 \quad \text{for } n = 1,2,\ldots,$$

so that

$$\int_0^T (\xi, v) \, dt \leq \liminf_{n \to +\infty} \int_0^T (\xi_n, w_n) \, dt.$$  

(6.8)

Next, multiply equations $w'_n - \kappa_n \Delta w_n + \xi_n = 0$ and $w' + \xi = 0$ by $w_n$ and $w$, respectively, and integrate them in time over $[0,T]$ to obtain

$$\frac{1}{2}|w_n(T)|^2_{L^2(\Omega)} + \kappa_n \int_0^T |\nabla w_n|^2_{L^2(\Omega)} + \int_0^T (\xi_n, w_n) \, dt = \frac{1}{2}|w_0|^2_{L^2(\Omega)}$$

and

$$\frac{1}{2}|w(T)|^2_{L^2(\Omega)} + \int_0^T (\xi, w) \, dt = \frac{1}{2}|w_0|^2_{L^2(\Omega)}.$$  

Hence

$$\limsup_{n \to +\infty} \int_0^T (\xi_n, w_n) \, dt \leq \frac{1}{2}|w_0|^2_{L^2(\Omega)} - \frac{1}{2}|w(T)|^2_{L^2(\Omega)} = \int_0^T (\xi, w) \, dt.$$  

(6.9)
Combining (6.8) and (6.9), we have
\[ \int_0^T (\xi, v-w) dt \leq 0 \]
for all \( v \in L^2(0,T;L^2(\Omega)) \) with \( f_a(u) \leq v \leq f_d(u) \) a.e. on \( Q_T \). This shows that \( \xi \in \partial I_u(w) \) in \( L^2(0,T;L^2(\Omega)) \), which is equivalent to (6.7). \( \square \)

References