1. Prove that

\[ 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1. \]

**Solution:** We give a proof by induction. For \( n = 0 \) the left hand side equals 1, and the right hand side equals \( 2^1 - 1 = 1 \), which proves the base case. Now suppose the equality holds for \( k \geq 1 \), and we show that it holds for \( k + 1 \). We have

\[
1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} \\
= 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1,
\]

which means that the equality holds when \( n = k + 1 \). \( \square \)

2. Prove or disprove:

\[ 2 + 4 + 6 + 8 + \cdots + 2n = (n - 1)(n + 2). \]

**Solution:** This is false, it should be \( n(n + 1) \) on the right hand side since the left hand side equals \( 2(1 + 2 + 3 + \cdots + n) = 2 \cdot \frac{n(n+1)}{2} \).

\( \square \)

3. Valid? Prove or disprove.

\[
\begin{align*}
(a) & \quad p \rightarrow q \\
   & \quad q \vee r \\
   & \quad r \rightarrow (\neg q) \\
(b) & \quad p \rightarrow q \\
   & \quad (\neg r) \vee (\neg q) \\
   & \quad r \\
   & \quad (\neg p)
\end{align*}
\]

**Solution:** Part (a) is invalid. When \( p, q, r \) are all true the assumptions are true but the conclusion is false. Part (b) is valid. We give a simple proof by contradiction. Suppose, seeking a contradiction, that the argument is invalid. Then, for some assignment the conclusion is false and the assumptions are all true. If the conclusion is false then \( p \) is true. By the first assumption and the fact that \( p \) is true we get that \( q \) is true. By the second assumption and the fact that \( q \) is true we get that \( r \) is false, but \( r \) is true by the third assumption, which is a contradiction.

\( \square \)
4. Valid? Prove or disprove.

If I work hard, then I earn lots of money.
If I don’t pay high taxes, then I don’t work hard.
If I work hard, then I pay high taxes.

Solution: The second assumption is the contrapositive of the conclusion, so the argument is clearly valid.

5. True or False questions.

(i) If $p \land q$ is true, then $p \lor q$ is true. TRUE
(ii) If $p \rightarrow q$ is true and $q \rightarrow p$ is true, then $p$ is logically equivalent to $q$. TRUE
(iii) If $\mathcal{A}$ is a tautology and $\mathcal{B}$ is a contradiction, then $\mathcal{A} \land (\neg \mathcal{B})$ is a tautology. TRUE
(iv) If $\mathcal{A} \iff \mathcal{B}$ and $\mathcal{C}$ is any statement, then $(\mathcal{A} \rightarrow \mathcal{C}) \iff (\mathcal{B} \rightarrow \mathcal{C})$. TRUE
(v) If the premises of an argument are all contradictions, then the argument is valid. TRUE
(vi) The statement $(p \rightarrow q) \iff (q \land (r \rightarrow s))$ evaluates to TRUE when all the atomic statements $p, q, r, s$ are true. TRUE

6. In the math department there are 30 personal computers (PCs).

- 20 have A drives,
- 8 have 19-inch monitors,
- 25 are running Windows XP,
- 20 have at least two of these properties,
- 6 have all three properties.

(a) How many PCs have at least one property?
(b) How many have none of these properties?
(c) How many have exactly one?

SOLUTION: The number of PCs that have at least one property can be calculated as follows. Let $A, B, C$ denote the sets of computers having A drives, 19-inch monitors, and those that are running Windows XP, respectively. Then the number PCs with at least one property is
the number of elements in the set $A \cup B \cup C$. Note that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. We have

$$|A \cup B \cup C| = |A| + |B \cup C| - |A \cap (B \cup C)|$$
$$= |A| + (|B| + |C| - |B \cap C|) - |(A \cap B) \cup (A \cap C)|$$
$$= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|)$$
$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

We already have $|A|, |B|, |C|$, and $|A \cap B \cap C|$. So, we just need to compute the number

$$|A \cap B| + |A \cap C| + |B \cap C|.$$

Note that this number is the number of elements in exactly 2 sets plus three times the number of elements in all 3 sets (draw a Venn diagram if you don’t see this immediately, or read the simple proof below), which can also be said to be the number of elements in at least 2 sets plus two times the number of elements in all 3 sets. Hence, this number is $20 + 2 \times 6 = 32$. So, the number of elements in at least one set is $20 + 8 + 25 - (32) + 6 = 27$.

Next, the number with none of the properties is the total number of computers minus the number that have at least one property, so $30 - 27 = 3$.

Finally the number with exactly one property is the number that have at least one property minus the number that have exactly two properties, so $27 - 20 = 7$.

**Claim:** For any finite sets $A, B, C$ contained in some universal finite set $U$, the number $|A \cap B| + |A \cap C| + |B \cap C|$ equals the number of elements that are in at least 2 of the sets plus two times the number of elements that are in all three sets. Proof: Denote by $S$ the elements of $U$ that are in at least 2 sets. We are asked to show that

$$|A \cap B| + |A \cap C| + |B \cap C| = |S| + 2|A \cap B \cap C|.$$
7. How many ways can you get a total of 6 when rolling two dice?

Solution: The possible ways to get 6 are if the die say (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), so there are 5 possibilities.

8. How many three digit numbers contain the digits 2 and 5 but not 0, 3, or 7?

Solution: I will consider 3 digit numbers in the range 000-999. In this case, 2 of the digits must be 2 and 5, and the last digit can be one of 1,2,4,5,6,8, or 9. There are 7 choices for the unknown digit, and there are several ways to permute the three digits once all are chosen. If the chosen digit is 2 or 5, then there are 3 permutations, and if the chosen digit is 1,4,6,8, or 9, there are 6 permutations. So, there are a total of $2 \times 3 + 5 \times 6 = 36$ possible numbers.

9. In a group of 29 people, how many people must there be whose birthdays are in the same month?

Solution: There are 12 months, so in a group of 29 people, by the Strong Pigeon Hole Principle, there are $\lceil \frac{29}{12} \rceil = \lceil 2.4167 \rceil = 3$ people who have birthdays in the same month.