THE $A$-POLYNOMIAL OF THE $(-2, 3, 3 + 2n)$ PRETZEL KNOTS

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Abstract. We show that the $A$-polynomial $A_n$ of the 1-parameter family of pretzel knots $K_n = (-2, 3, 3 + 2n)$ satisfies a linear recursion relation of order 4 with explicit constant coefficients and initial conditions. Our proof combines results of Tamura-Yokota and the second author. As a corollary, we show that the $A$-polynomial of $K_n$ and the mirror of $K_{-n}$ are related by an explicit $GL(2, \mathbb{Z})$ action. We leave open the question of whether or not this action lifts to the quantum level.

1. Introduction

1.1. The behavior of the $A$-polynomial under filling. In [CCG+94], the authors introduced the $A$-polynomial $A_W$ of a hyperbolic 3-manifold $W$ with one cusp. It is a 2-variable polynomial which describes the dependence of the eigenvalues of a meridian and longitude under any representation of $\pi_1(W)$ into $SL(2, \mathbb{C})$. The $A$-polynomial plays a key role in two problems:

- the deformation of the hyperbolic structure of $W$,
- the problem of exceptional (i.e., non-hyperbolic) fillings of $W$.

Knowledge of the $A$-polynomial (and often, of its Newton polygon) is translated directly into information about the above problems, and vice-versa. In particular, as demonstrated by Boyer and Zhang [BZ01], the Newton polygon is dual to the fundamental polygon of the Culler-Shalen seminorm [CGLS87] and, therefore, can be used to classify cyclic and finite exceptional surgeries.

In [Gar10], the first author observed a pattern in the behavior of the $A$-polynomial (and its Newton polygon) of a 1-parameter family of 3-manifolds obtained by fillings of a 2-cusped manifold. To state the pattern, we need to introduce some notation. Let $K = \mathbb{Q}(x_1, \ldots, x_r)$ denote the field of rational functions in $r$ variables $x_1, \ldots, x_r$.

Definition 1.1. We say that a sequence of rational functions $R_n \in K$ (defined for all integers $n$) is holonomic if it satisfies a linear recursion with constant coefficients. In other words, there exists a natural number $d$ and $c_k \in K$ for $k = 0, \ldots, d$ with $c_d \neq 0$ such that for all integers $n$ we have:

$$\sum_{k=0}^{d} c_k R_{n+k} = 0 \tag{1}$$

Depending on the circumstances, one can restrict attention to sequences indexed by the natural numbers (rather than the integers).

Consider a hyperbolic manifold $W$ with two cusps $C_1$ and $C_2$. Let $(\mu_i, \lambda_i)$ for $i = 1, 2$ be pairs of meridian-longitude curves, and let $W_n$ denote the result of $-1/n$ filling on $C_2$. Let $A_n(M_1, L_1)$ denote the $A$-polynomial of $W_n$ with the meridian-longitude pair inherited from $W$.

Theorem 1.1. [Gar10] With the above conventions, there exists a holonomic sequence $R_n(M_1, L_1) \in \mathbb{Q}(M_1, L_1)$ such that for all but finitely many integers $n$, $A_n(M_1, L_1)$ divides the numerator of $R_n(M_1, L_1)$. In addition, a recursion for $R_n$ can be computed explicitly via elimination, from an ideal triangulation of $W$.

Date: April 8, 2011.
S.G. was supported in part by NSF.

Key words and phrases: pretzel knots, $A$-polynomial, Newton polygon, character variety, Culler-Shalen seminorm, holonomic sequences, quasi-polynomials.
1.2. The Newton polytope of a holonomic sequence. Theorem 1.1 motivates us to study the Newton polytope of a holonomic sequence of Laurent polynomials. To state our result, we need some definitions. Recall that the Newton polytope of a holonomic sequence of Laurent polynomials in $n$ variables $x_1, \ldots, x_n$ is the convex hull of the points whose coordinates are the exponents of its monomials. Recall that a quasi-polynomial is a function $p : \mathbb{N} \rightarrow \mathbb{Q}$ of the form $p(n) = \sum_{k=0}^{d} c_k(n)n^k$ where $c_k : \mathbb{N} \rightarrow \mathbb{Q}$ are periodic functions. When $c_d \neq 0$, we call $d$ the degree of $p(n)$. We will call quasi-polynomials of degree at most one (resp. two) quasi-linear (resp. quasi-quadratic). Quasi-polynomials appear in lattice point counting problems (see [Ehr62, CW10]), in the Slope Conjecture in quantum topology (see [Gar11b]), in enumerative combinatorics (see [Gar11a]) and also in the A-polynomial of filling families of 3-manifolds (see [Gar10]).

Definition 1.2. We say that a sequence $N_n$ of polytopes is linear (resp. quasi-linear) if the coordinates of the vertices of $N_n$ are polynomials (resp. quasi-polynomials) in $n$ of degree at most one. Likewise, we say that a sequence $N_n$ of polytopes is quadratic (resp. quasi-quadratic) if the coordinates of the vertices of $N_n$ are polynomials (resp. quasi-polynomials) of degree at most two.

Theorem 1.2. [Gar10] Let $N_n$ be the Newton polytope of a holonomic sequence $R_n \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$. Then, for all but finitely many $n$, $N_n$ is quasi-linear.

1.3. Do favorable links exist? Theorems 1.1 and 1.2 are general, but in favorable circumstances more is true. Namely, consider a family of knot complements $K_n$, obtained by $-1/n$ filling on a cusp of a 2-component hyperbolic link $J$. Let $f$ denote the linking number of the two components of $J$, and let $A_n$ denote the A-polynomial of $K_n$ with respect to its canonical meridian and longitude $(M, L)$. By definition, $A_n$ contains all components of irreducible representations, but not the component $L - 1$ of abelian representations.

Definition 1.3. We say that $J$, a 2-component link in 3-space, with linking number $f$ is favorable if $A_n(M, LM^{-f^2n}) \in \mathbb{Q}[M^{\pm 1}, L^{\pm 1}]$ is holonomic.

The shift of coordinates, $LM^{-f^2n}$, above is due to the canonical meridian-longitude pair of $K_n$ differing from the corresponding pair for the unfilled component of $J$ as a result of the nonzero linking number. Theorem 1.2 combined with the above shift implies that, for a favorable link, the Newton polygon of $K_n$ is quasi-quadratic.

Hoste-Shanahan studied the first examples of a favorable link, the Whitehead link and its half-twisted version (see Figure 1), and consequently gave an explicit recursion relation for the 1-parameter families of A-polynomials of twist knots $K_{2,n}$ and $K_{3,n}$ respectively; see [HS04].

![Figure 1](image-url) The Whitehead link on the left, the half-twisted Whitehead link in the middle and our seed link $J$ at right.

The goal of our paper is to give another example of a favorable link $J$ (see Figure 1), whose 1-parameter filling gives rise to the family of $(-2, 3, 3 + 2n)$ pretzel knots. Our paper is a concrete illustration of the general Theorems 1.1 and 1.2 above. Aside from this, the 1-parameter family of knots $K_n$, where $K_n$ is the $(-2, 3, 3 + 2n)$ pretzel knot, is well-studied in hyperbolic geometry (where $K_n$ and the mirror of $K_{-n}$ are pairs of geometrically similar knots; see [BH96, MM08]), in exceptional Dehn surgery (where for instance...
$K_2 = (-2, 3, 7)$ has three Lens space fillings 1/0, 18/1 and 19/1; see [CGLS87]) and in Quantum Topology (where $K_n$ and the mirror of $K_{-n}$ have different Kashaev invariant, equal volume, and different subleading corrections to the volume, see [GZ]).

The success of Theorems 1.3 and 1.4 below hinges on two independent results of Tamura-Yokota and the second author [TY04, Mat02], and an additional lucky coincidence. Tamura-Yokota compute an explicit recursion relation, as in Theorem 1.3, by elimination, using the gluing equations of the decomposition of the complement of $J$ into six ideal tetrahedra; see [TY04]. The second author computes the Newton polygon $N_n$ of the $A$-polynomial of the family $K_n$ of pretzel knots; see [Mat02]. This part is considerably more difficult, and requires:

(a) The set of boundary slopes of $K_n$, which are available by applying the Hatcher-Oertel algorithm [HO89, Dun01] to the 1-parameter family $K_n$ of Montesinos knots. The four slopes given by the algorithm are candidates for the slopes of the sides of $N_n$. Similarly, the fundamental polygon of the Culler-Shalen seminorm of $K_n$ has vertices in rays which are the multiples of the slopes of $N_n$. Taking advantage of the duality of the fundamental polygon and Newton polygon, in order to describe $N_n$ it is enough to determine the vertices of the Culler-Shalen polygon.

(b) Use of the exceptional 1/0 filling and two fortunate exceptional Seifert fillings of $K_n$ with slopes $4n + 10$ and $4n + 11$ to determine exactly the vertices of the Culler-Shalen polygon and consequently $N_n$. In particular, the boundary slope 0 is not a side of $N_n$ (unless $n = -3$) and the Newton polygon is a hexagon for all hyperbolic $K_n$.

Given the work of [TY04] and [Mat02], if one is lucky enough to match $N_n$ of [Mat02] with the Newton polygon of the solution of the recursion relation of [TY04] (and also match a leading coefficient), then Theorem 1.3 below follows; i.e., $J$ is a favorable link.

1.4. Our results for the pretzel knots $K_n$. Let $A_n(M, L)$ denote the $A$-polynomial of the pretzel knot $K_n$, using the canonical meridian-longitude coordinates. Consider the sequences of Laurent polynomials $P_n(M, L)$ and $Q_n(M, L)$ defined by:

\[(2)\quad P_n(M, L) = A_n(M, LM^{-4n})\]

for $n > 1$ and

\[(3)\quad Q_n(M, L) = A_n(M, LM^{-4n})M^{-4(3n^2 + 11n + 4)}\]

for $n < -2$ and $Q_{-2}(M, L) = A_{-2}(M, LM^{-8})M^{-20}$. In the remaining cases $n = -1, 0, 1$, the knot $K_n$ is not hyperbolic (it is the torus knot $5_1$, $8_{19}$ and $10_{124}$ respectively), and one expects exceptional behavior. This is reflected in the fact that $P_n$ for $n = 0, 1$ and $Q_n$ for $n = -1, 0$ can be defined to be suitable rational functions (rather than polynomials) of $M, L$. Let $NP_n$ and $NQ_n$ denote the Newton polygons of $P_n$ and $Q_n$ respectively.

**Theorem 1.3.** (a) $P_n$ and $Q_n$ satisfy linear recursion relations

\[(4)\quad \sum_{k=0}^{4} c_k P_{n+k} = 0, \quad n \geq 0\]

and

\[(5)\quad \sum_{k=0}^{4} c_k Q_{n-k} = 0, \quad n \leq 0\]

where the coefficients $c_k$ and the initial conditions $P_n$ for $n = 0, \ldots, 3$ and $Q_n$ for $n = -3, \ldots, 0$ are given in Appendix A.

(b) In $(L, M)$ coordinates, $NP_n$ and $NQ_n$ are hexagons with vertices

\[(6)\quad \{0, 0\}, \{1, -4n + 16\}, \{n - 1, 12n - 12\}, \{2n + 1, 16n + 18\}, \{3n - 1, 32n - 10\}, \{3n, 28n + 6\}\]

for $P_n$ with $n > 1$ and

\[(7)\quad \{0, 4n + 28\}, \{1, 38\}, \{-n, -12n + 26\}, \{-2n - 3, -16n - 4\}, \{-3n - 4, -28n - 16\}, \{-3n - 3, -32n - 6\}\]
for $Q_n$ with $n < -1$.

**Remark 1.4.** We can give a single recursion relation valid for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ as follows. Define

$$R_n(M, L) = A_n(M, LM^{-4n}) b^{|n|} \epsilon_n(M),$$

where

$$b = \frac{1}{LM^8(1 - M^2)(1 + LM^{-10})}, \quad c = \frac{M^3 M^{12}(1 - M^2)^3}{(1 + LM^{-10})^3} \quad \epsilon_n(M) = \begin{cases} 1 & \text{if } n > 1 \\ cM^{-4(3+n)(2+3n)} & \text{if } n < -2 \\ cM^{-28} & \text{if } n = -2 \end{cases}$$

Then, $R_n$ satisfies the palindromic fourth order linear recursion

$$\sum_{k=0}^{4} \gamma_k R_{n+k} = 0$$

where the coefficients $\gamma_k$ and the initial conditions $R_n$ for $n = 0, \ldots, 3$ are given in Appendix B. Moreover, $R_n$ is related to $P_n$ and $Q_n$ by:

$$R_n = \begin{cases} P_n b^{|n|} & \text{if } n \geq 0 \\ Q_n b^{|n|} cM^{-8} & \text{if } n \leq 0 \end{cases}$$

**Remark 1.5.** The computation of the Culler-Shalen seminorm of the pretzel knots $K_n$ has an additional application, namely it determines the number of components (containing the character of an irreducible representation) of the SL(2, $\mathbb{C}$) character variety of the knot, and consequently the number of factors of its $A$-polynomial. In the case of $K_n$, (after translating the results of [Mat02] for the pretzel knots $(-2, 3, n)$ to the pretzel knots $(-2, 3, 3 + 2n)$) it was shown by the second author [Mat02, Theorem 1.6] that the character variety of $K_n$ has one (resp. two) components when 3 does not divide $n$ (resp. divides $n$). The non-geometric factor of $A_n$ is given by

$$\begin{cases} 1 - LM^{4(n+3)} & n \geq 3 \\ L - M^{-4(n+3)} & n \leq -3 \end{cases}$$

for $n \neq 0$ a multiple of 3.

Since the $A$-polynomial has even powers of $M$, we can define the $B$-polynomial by

$$B(M^2, L) = A(M, L).$$

Our next result relates the $A$-polynomials of the geometrically similar pair $(K_n, -K_{-n})$ by an explicit GL(2, $\mathbb{Z}$) transformation.
Theorem 1.4. For \( n > 1 \) we have:
\[
B_n(M, LM^{2n-5}) = (-L)^n M^{8(2n^2-7n+7)} B_n(-L^{-1}, L^{2n+5} M^{-1}) \eta_n
\]
where \( \eta_n = 1 \) (resp. \( M^{22} \)) when \( n > 2 \) (resp. \( n = 2 \)).

2. Proofs

2.1. The equivalence of Theorem 1.3 and Remark 1.4. In this subsection we will show the equivalence of Theorem 1.3 and Remark 1.4. Let \( \gamma_k = c_k/b^k \) for \( k = 0, \ldots, 4 \) where \( b \) is given by (9). It is easy to see that the \( \gamma_k \) are given explicitly by Appendix B, and moreover, they are palindromic. Since \( R_n = P_n b^n \) for \( n = 0, \ldots, 3 \) it follows that \( R_n \) and \( P_n b^n \) satisfy the same recursion relation (10) for \( n \geq 0 \) with the same initial conditions. It follows that \( R_n = P_n b^n \) for \( n \geq 0 \).

Solving (10) backwards, we can check by an explicit calculation that \( R_n = Q_n b^n c M^{-8} \) for \( n = -3, \ldots, 0 \) where \( b \) and \( c \) are given by (9). Moreover, \( R_n \) and \( Q_n b^n c M^{-8} \) satisfy the same recursion relation (10) for \( n < 0 \). It follows that \( R_n = Q_n b^n c M^{-8} \) for \( n < 0 \). This concludes the proof of Equations (10) and (11).

2.2. Proof of Theorem 1.3. Let us consider first the case of \( n \geq 0 \), and denote by \( P'_n \) for \( n \geq 0 \) the unique solution to the linear recursion relation (4) with the initial conditions as in Theorem 1.3. Let \( R'_n = P'_n b^n \) be defined according to Equation (11) for \( n \geq 0 \).

Remark 1.4 implies that \( R'_n \) satisfies the recursion relation of [TY04, Thm.1]. It follows by [TY04, Thm.1] that \( A_n(M, LM^{-4n}) \) divides \( P'_n(M, L) \) when \( n > 1 \).

Next, we claim that the Newton polygon \( N P'_n \) of \( P'_n(M, L) \) is given by (6). This can be verified easily by induction on \( n \).

Next, in [Mat02, p.1286], the second author computes the Newton polygon \( N_n \) of the \( A_n(M, L) \). It is a hexagon in \( (L, M) \) coordinates by
\[
\{\{0,0\}, \{1,16\}, \{n-1,4(n^2+2n-3)\}, \{2n+1,2(4n^2+10n+9)\},
\{3n-1,2(6n^2+14n-5)\}, \{3n,2(6n^2+14n+3)\}\}
\]
when \( n > 1 \),
\[
\{\{-3n-4,0\}, \{-3(1+n),10\}, \{-3-2n,4(3+4n+n^2)\},
\{-n,2(4n^2+16n+21)\}, \{0,4(3n^2+12n+11)\}, \{1,6(2n^2+8n+9)\}\}
\]
when \( n < -2 \) and
\[
\{\{0,0\}, \{1,0\}, \{2,4\}, \{1,10\}, \{2,14\}, \{3,14\}\}
\]
when \( n = -2 \). Notice that the above 1-parameter families of Newton polygons are quadratic. It follows by explicit calculation that the Newton polygon of \( A_n(M, LM^{-4n}) \) is quadratic and exactly agrees with \( N P'_n \) for all \( n > 1 \).

The above discussion implies that \( P_n(M, L) \) is a rational multiple of \( A_n(M, LM^{-4n}) \). Since their leading coefficients (with respect to \( L \)) agree, they are equal. This proves Theorem 1.3 for \( n > 1 \). The case of \( n < -1 \) is similar.

2.3. Proof of Theorem 1.4. Using Equations (2) and (3), convert Equation (12) into
\[
Q_{-n}(\sqrt{M}, L/M^5) = (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M).
\]
Note that, under the substitution \((M, L) \mapsto (i/\sqrt{L}, L^{2n+5}/M)\), \( LM^{4n} \) becomes \( L^5/M \). Similarly, \( LM^{-4n} \) becomes \( L/M^{5} \) under the substitution \((M, L) \mapsto (\sqrt{M}, LM^{2n-5})\).

It is straightforward to verify equation (13) for \( n = 2, 3, 4, 5 \). For \( n \geq 6 \), we use induction. Let \( c_k \) denote the result of applying the substitutions \((M, L) \mapsto (\sqrt{M}, L/M^5)\) to the \( c_k \) coefficients in the recursions (4) and (5). For example,
\[
c_0 = L^4(1+L)^4(1-M)^4/M^2.
\]
Similarly, define $c_k^+$ to be the result of the substitution $(M, L) \mapsto (i/\sqrt{L}, L^5/M)$ to $c_k$. It is easy to verify that for $k = 0, 1, 2, 3,$

$$\frac{c_k^-}{c_4} (-LM)^{k-4} = \frac{c_k^+}{c_4}.$$ 

Then,

$$Q_n(\sqrt{M}, L/M^5) = -\frac{1}{c_4} \sum_{k=0}^{3} c_k^- Q_{n+k-4}(\sqrt{M}, L/M^5)$$

$$= -\frac{1}{c_4} \sum_{k=0}^{3} c_k^- (-L)^{n-4+k} M^{n-4+k+13} P_{n-4+k}(i\sqrt{L}, L^5/M)$$

$$= -(-L)^n M^{n+13} \sum_{k=0}^{3} \frac{c_k^-}{c_4} (-LM)^{k-4} P_{n-4+k}(i\sqrt{L}, L^5/M)$$

$$= -(-L)^n M^{n+13} \sum_{k=0}^{3} \frac{c_k^+}{c_4} P_{n-4+k}(i\sqrt{L}, L^5/M)$$

$$= (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M).$$

By induction, equation (13) holds for all $n > 1$ proving Theorem 1.4. \(\square\)

**Appendix A. The coefficients $c_k$ and the initial conditions for $P_n$ and $Q_n$**

$$c_4 = M^4$$

$$c_3 = 1 + M^4 + 2LM^{12} + LM^{14} - LM^{16} + L^2 M^{20} - L^2 M^{22} - 2L^2 M^{24} - L^3 M^{32} - L^3 M^{36}$$

$$c_2 = (-1 + LM^{12}) (-1 - 2LM^{10} - 3LM^{12} + 2LM^{14} - L^2 M^{16} + 2L^2 M^{18} - 4L^2 M^{20} - 2L^2 M^{22} + 3L^2 M^{24} - 3L^3 M^{28} + 2L^3 M^{30} + 4L^3 M^{32} - 2L^3 M^{34} + L^3 M^{36} - 2L^4 M^{38} + 3L^4 M^{40} + 2L^4 M^{42} + L^5 M^{52})$$

$$c_1 = -L^2 (-1 + M)^2 M^{16} (1 + M) (1 + LM^{10}) (1 - M^4 - 2LM^{12} - LM^{14} + LM^{16} - L^2 M^{20} + L^2 M^{22} + 2L^2 M^{24} + L^3 M^{32} + L^3 M^{36})$$

$$c_0 = L^4 (-1 + M)^4 M^{56} (1 + M)^4 (1 + LM^{10})^4$$

$$P_0 = \frac{(-1 + LM^{12}) (1 + LM^{12})^2}{(1 + LM^{10})^4}$$

$$P_1 = \frac{(-1 + LM^{11})^2 (1 + LM^{11})^2}{1 + LM^{10}}$$

$$P_2 = -1 + LM^8 - 2LM^{10} + LM^{12} + 2L^2 M^{20} + L^2 M^{22} - L^4 M^{40} - 2L^4 M^{42} - L^5 M^{50} + 2L^5 M^{52} - L^5 M^{54} + L^6 M^{62}$$

$$P_3 = (-1 + LM^{12}) (-1 + LM^4 - LM^6 + 5LM^{10} + LM^{12} + 5L^2 M^{16} - 4L^2 M^{18} + L^3 M^{22} + L^3 M^{26} + 3L^3 M^{30} + 2L^3 M^{32} - 2L^4 M^{36} + 3L^4 M^{38} + 3L^4 M^{40} + 2L^4 M^{42} + 2L^5 M^{46} - 2L^5 M^{48} + 3L^5 M^{50} + 2L^5 M^{52} - L^6 M^{56} + 2L^6 M^{60} - 5L^6 M^{62} - L^7 M^{66} + 5L^7 M^{68} - 2L^7 M^{70} + L^7 M^{72} - L^7 M^{74} + L^8 M^{78})$$
The Culler-Shalen seminorms of the knot

A. Hatcher and U. Oertel, Boundary slopes for Montesinos knots

Jim Hoste and Patrick D. Shanahan, A formula for the A-polynomial of twist knots

Let $P_n$ be as in Appendix A. Then,

$$R_n = P_n b^n$$

for $n = 0, \ldots, 3$ where $b$ is given by Equation (9).

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