Math. 2403, Practice Test2, Solutions

1. Use the method of undetermined coefficients or variation of parameters to solve the initial value problem

\[ y'' - y = -4xe^{-x}, \quad y(0) = 0, \quad y'(0) = 1. \]

A general solution of the homogeneous equation is \( y_h(x) = c_1e^x + c_2e^{-x} \). We now use the method of undetermined coefficients and look for a particular solution of the form \( y_p(x) = x^s(Ax + B)e^{-x} \). Since \( Be^{-x} \) is a solution of the homogeneous equation we take \( s = 1 \). Then \( y_p(x) = (Ax^2 + Bx)e^{-x} \) and substituting \( y_p \) for \( y \) in the equation we get

\[-4Axe^{-x} + (2A - 2B)e^{-x} = -4xe^{-x}\]

which gives \( A = 1, B = 1 \). Therefore the general solution is

\[ y(x) = c_1e^x + c_2e^{-x} + (x^2 + x)e^{-x}. \]

We now need to find \( c_1 \) and \( c_2 \). We must have

\[ 0 = c_1 + c_2, \quad 1 = c_1 - c_2 + 1 \]

and so \( c_1 = c_2 = 0 \). Therefore finally

\[ y(x) = (x^2 + x)e^{-x}. \]

2. An 8-kg mass is attached to a spring hanging from the ceiling, thereby causing the spring to stretch 1.96 m upon coming to rest at equilibrium. At time \( t = 0 \), an external force \( f(t) = \cos 2t \) N is applied to the system. The damping constant for the system is 3 N-sec/m. Determine the steady state solution of the system (i.e. the part of the solution that is left for big \( t \) after the transient solution has been eliminated.) Assume that \( g = 9.8 \) m/sec\(^2\).

We first calculate the spring constant \( k \). We must have \( 8 \cdot 9.8 = k \cdot 1.96 \) which gives \( k = 40 \). Therefore the equation of motion is

\[ 8x'' + 3x' + 40x = \cos 2t. \]

The characteristic polynomial has roots

\[ r_{\pm} = \frac{-3 \pm \sqrt{9 - 4 \cdot 8 \cdot 40}}{16} \]

that both have negative real parts. Therefore

\[ \lim_{t \to \infty} x_h(t) = 0. \]
We now look for \( x_p(t) = A \cos 2t + B \sin 2t \). Computing the derivatives of \( x_p \) and plugging them into the equation we obtain
\[
8(-4A \cos 2t - 4B \sin 2t) + 3(-2A \cos 2t + 2B \sin 2t) + 40(A \cos 2t + B \sin 2t) = \cos 2t
\]
which produces a system
\[
8A + 6B = 1, \quad 8B - 6A = 0.
\]
Solving it we get \( A = 0.08, B = 0.06 \). Therefore a general solution is
\[
x(t) = x_h(t) + 0.08 \cos 2t + 0.06 \sin 2t
\]
and since the homogeneous part \( x_h(t) \) goes to 0 as \( t \to \infty \) (it is the transient solution) the steady state solution is equal to \( x_p(t) = 0.08 \cos 2t + 0.06 \sin 2t \). Notice that the amplitude of the steady state solution is equal to \( C = \sqrt{0.08^2 + 0.06^2} = 0.1 \) and the phase angle is equal to \( \alpha = \arctan(0.06/0.08) = \arctan(3/4) \). Therefore we can rewrite the above function as \( 0.1 \cos(2t - \arctan(3/4)) \).

3.(a) Is the function
\[
f(t) = e^{t^2/2}.
\]
of exponential order \( \alpha \) for some \( \alpha \)? If yes, find the range of \( \alpha \).

Since \( t^2/(t + 1) < t \) for \( t > 0 \) we have
\[
e^{t^2/2} < e^{\alpha t}
\]
for every \( \alpha \geq 1 \). On the other hand if \( \alpha < 1 \) then
\[
\lim_{t \to +\infty} \left( \frac{t^2}{t + 1} - \alpha t \right) \to +\infty
\]
and so \( f(t) \) is not of exponential order \( \alpha \) for \( \alpha < 1 \). Therefore \( f(t) \) is of exponential order \( \alpha \) for \( \alpha \geq 1 \).

(b) Find \( L\{f(t)\} \) if
\[
f(t) = \begin{cases} 
e^{2t} & \text{if } 0 < t < 3 \\
1 & \text{if } t > 3.
\end{cases}
\]
\[
L\{f\}(s) = \int_0^3 e^{-st} e^{2t} dt + \int_3^\infty e^{-st} dt = \frac{1}{2-s} e^{t(2-s)} \bigg|_0^3 + \frac{e^{-3s}}{s}
\]
\[
= \frac{1}{2-s} (e^{3(2-s)} - 1) + \frac{e^{-3s}}{s} \quad \text{for } s > 2.
\]

4. Find \( L^{-1}\{F(s)\} \) if
\[
F(s) = \frac{3s + 5}{s^2 - 2s + 5}.
\]
\[ \frac{3s + 5}{s^2 - 2s + 5} = \frac{3(s - 1)}{(s - 1)^2 + 4} + \frac{8}{(s - 1)^2 + 4}. \]

Therefore

\[ L^{-1}\{F(s)\} = 3e^t \cos 2t + 4e^t \sin 2t. \]

5. Use Laplace transform to solve the initial value problem

\[ y''' + y'' + y' + y = 1, \quad y(0) = y'(0) = 0, y''(0) = 1. \]

Taking Laplace transform of the equation we obtain

\[ s^3Y(s) - 1 + s^2Y(s) + sY(s) + Y(s) = \frac{1}{s}. \]

which gives us

\[ Y(s) = \frac{1 + s}{s(s^3 + s^2 + s + 1)} = \frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}. \]

Therefore, taking inverse Laplace transform, we obtain

\[ y(t) = 1 - \cos t. \]