Practice Test 2

1. Find the entropy solution of the following initial value problem. Draw a picture depicting what happens for all times $t \geq 0$ that include characteristics and the shock curve.

$$\begin{cases} u_t + \frac{1}{2}(e^{u^2})_x = 0 \\ u(0, x) = \begin{cases} 0 & x > 1 \\ 1 & x < 1. \end{cases} \end{cases}$$

2. Find the continuous weak solution of the initial value problem

$$\begin{cases} u_t + (\ln u)_x = 0 \\ u(0, x) = \begin{cases} 1 & x > 0 \\ 2 & x < 0. \end{cases} \end{cases}$$

3. Consider the Cauchy problem

$$\begin{cases} u_t + u^2u_x + u^3 = x^3, & t > 0, x \in \mathbb{R} \\ u(0, x) = \cos x, & x \in \mathbb{R}. \end{cases}$$

Verify that the assumptions of the Cauchy-Kovalevskaya theorem are satisfied and then find the terms of order $\leq 3$ of the Taylor series of the solution about the origin.

4. Determine the sets of points where the equation

$$x^2u_{xx} - 4xu_{xy} + y^2u_{yy} = 0$$

is elliptic, hyperbolic, and parabolic. Find the vectors characteristic with respect to the operator given by the above equation at $(x, y) = (1, 0)$. 
Answers

1. \[ u(t, x) = \begin{cases} 0 & x > \frac{e - 1}{2} t + 1 \\ 1 & x < \frac{e - 1}{2} t + 1 \end{cases} \]

2. \[ u(t, x) = \begin{cases} 1 & x > t \\ \frac{t}{x} & \frac{t}{2} \leq x \leq t \\ 2 & x < \frac{t}{2} \end{cases} \]

3. \[ u(t, x) = 1 - t - \frac{1}{2} x^2 + tx + t^2 + \frac{3}{2} x^2 t - 4 t x^2 + \text{terms of order } \geq 4 \]

4. Equation is elliptic if \(|y| > 2\), parabolic if \(y = \pm 2\), and hyperbolic if \(|y| < 2\). The vectors characteristic with respect to the operator at \((x, y) = (1, 0)\) are

\[ \vec{\xi} = \begin{pmatrix} 0 \\ t \end{pmatrix} \quad \text{or} \quad \vec{\xi} = \begin{pmatrix} 4t \\ t \end{pmatrix} \quad t \in \mathbb{R}. \]