1. (a) \[ \tilde{f}(x) \]

\[ \tilde{f}(x) \]

(b) \( f \) is odd \( \Rightarrow a_n = 0 \) for \( n = 0, 1, 2, \ldots \)

\[ b_n = 2 \int_0^1 (1-x) \sin \pi n x \, dx = \frac{2}{\pi n} \left( 1 - x \right) \left( -\cos \pi n x \right) \bigg|_0^1 - \frac{2}{\pi n} \int_0^1 \cos \pi n x \, dx \]

\[ = \frac{2}{\pi n} - \frac{2}{(\pi n)^2} \sin \pi n x \bigg|_0^1 = \boxed{\frac{2}{\pi n}} \quad \text{for} \quad n = 1, 2, 3, \ldots \]

(c) The differentiated Fourier series for \( f \) is

\[ \sum_{n=1}^{\infty} 2 \cos \pi n x \]

which is divergent and so it does not converge to \( [\tilde{f}(x)]' \).

(Notice that \( f \) is not continuous so no convergence theorem applies.)

2. The family is not orthonormal since it is not orthogonal, for instance \( (1, \sqrt{2} \sin \pi n x) \neq 0 \).

\[ (1 + \cos \pi n x) \sin x = \sin \pi n x + \frac{1}{2} \sin 2\pi n x \] and therefore it belongs to the space spanned by the family.
(3) (a) \( \lambda < 0 \), \( \lambda = -\beta^2 \), \( \beta > 0 \). Then \( u(x) = Ae^{-\beta x} + Be^{\beta x} \)

\[ u'(x) = \beta Ae^{-\beta x} - \beta Be^{\beta x} \]

\[
\begin{cases}
A - B = 0 \\
A e^{\beta \frac{\pi}{2}} - B e^{-\beta \frac{\pi}{2}} = 0
\end{cases} \Rightarrow A = B = 0 \quad \text{No negative eigenvalues}
\]

\( \lambda = 0 \)

\( u(x) = Ax + B \) and \( |u(x)| = 1 \) is an eigenvector

\( \lambda > 0 \), \( \lambda = \beta^2, \beta > 0 \)

Then \( u(x) = A \cos \beta x + B \sin \beta x \), \( u'(x) = -A \beta \sin \beta x + B \beta \cos \beta x \)

\[
\begin{cases}
B = 0 \\
-A \beta \sin \beta \frac{\pi}{2} + B \beta \cos \beta \frac{\pi}{2} = 0
\end{cases} \Rightarrow \frac{B}{2} = u \frac{\pi}{2} \iff \beta = 2n, n = 1, 2, \ldots
\]

\( u_n(x) = \cos 2nx \)

are eigenvectors for eigenvalues \( \lambda_n = 4n^2, n = 1, 2, \ldots \)

(b) The two smallest eigenvalues are 0 and 4 and so (in the current notation) \( u_1(x) = 1 \), \( u_2(x) = \cos 2x \).

\[
\text{Proj} \left( f; \text{span} \{u_1, u_2\} \right) = \left( \frac{\sin 2x, u_1}{\|u_1\|^2} \right) u_1 + \left( \frac{\sin 2x, u_2}{\|u_2\|^2} \right) u_2
\]

\[
= \frac{\int_0^{\pi/2} \sin 2x \ dx}{\pi/2} + \frac{\int_0^{\pi/2} \sin 2x \cos 2x \ dx}{\cos 2x}
\]

\[
= \left. -\cos 2x \right|_0^{\pi/2} + \frac{1}{2} \frac{\int_0^{\pi/2} \sin 4x \ dx}{\cos 2x} = 2 \cdot \frac{2}{\pi} + 0
\]

\[= \frac{2}{\pi} \quad \text{so the projection is the constant function } g(x) = \frac{2}{\pi} . \]