1. Solve the following boundary value problem:

\[
\begin{align*}
\n \nabla^2 u(x, y) &= x \cos \pi y \quad \text{for } 0 < x, y < 1, \\
 u(0, y) &= u(1, y) = 0 \quad \text{for } 0 < y < 1, \\
 u_y(x, 0) &= \sin \pi x, \quad u_y(x, 1) = 0 \quad \text{for } 0 < x < 1.
\end{align*}
\]

2 subproblems:

\(1\) \( \begin{cases} 
\nabla^2 u = 0 \\
 u(0, y) = u(1, y) = 0 \\
 u_y(x, 0) = \sin \pi x, \quad u_y(x, 1) = 0 
\end{cases} \)

\(2\) \( \begin{cases} 
\nabla^2 u = x \cos \pi y \\
 u(0, y) = u(1, y) = 0 \\
 u_y(x, 0) = u_y(x, 1) = 0 
\end{cases} \)

\(1\) \( u(x, y) = X(x) Y(y). \) Then

\(-X''(x) = \lambda X(x), \quad X(0) = X(1) = 0 \)

\( Y''(y) = \lambda Y(y), \quad Y'(1) = 0 \)

Solving the SL problem we get \( \lambda_n = (n\pi)^2 \), \( X_n(x) = \sin n\pi x \)

for \( n = 1, 2, 3, \ldots \)

\( Y_n(y) = A_n \cosh (y-1)n\pi \)

\( u(x, y) = \sum_{n=1}^{\infty} A_n \cosh (y-1)n\pi \sin n\pi x \) solves the homogeneous part of \(1\).

\( u_y(x, 0) = \sum_{n=1}^{\infty} A_n \sinh (-n\pi) \sin n\pi x = \sin \pi x \)

Since \( \sin n\pi x \) is equal to its expansion in terms of \( \{ \sin n\pi x \}_{n=1}^{\infty} \),
we obtain \( A_n = 0 \) for \( n=2, 3, \ldots \), and

\( A_1 = \frac{-1}{\pi \sinh \pi} \)

\( u_1(x, y) = \frac{-1}{\pi \sinh \pi} \cosh \pi(y-1) \sin n\pi x \) solves \(1\).
Orthogonal bases of eigenfunctions associated with homogeneous boundary conditions are \( \{ \sin n \pi x \}, \ n = 1, 2, \ldots \), and \( \{ \cos m \pi y \}, \ m = 0, 1, 2, \ldots \).

Therefore we will look for a solution in the form
\[
U(x, y) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} U_{nm} \sin n \pi x \cos m \pi y
\]

\[
\frac{\partial^2 U(x, y)}{\partial x^2} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( -n^2 - m^2 \right) \pi^2 \sin n \pi x \cos m \pi y
\]

Now expand:
\[
x \cos m \pi y = 4 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (cos n \pi x, \ sin n \pi x \ cos m \pi y) \ sin n \pi x \ cos m \pi y
\]

\[(cos n \pi x, \ sin n \pi x \ cos m \pi y) = \int_0^1 \int_0^1 x \ cos m \pi y \ sin n \pi x \ cos m \pi y \ dx \ dy
\]

\[
= \begin{cases} 
0 & \text{if } m \neq 1 \\
\frac{(-1)^{n+1}}{2n \pi} & \text{if } m = 1 
\end{cases}
\]

Therefore we get from the equation
\[
\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} U_{nm} (-n^2 - m^2) \pi^2 \sin n \pi x \cos m \pi y = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n \pi} \sin n \pi x \cos m \pi y
\]

\[
\therefore \ U_{nm} = 0 \begin{cases} 
0 & \text{if } m \neq 1 \\
\frac{2 (-1)^n}{n \pi \left( \frac{1}{3} n^2 +1 \right)} & \text{if } m = 1 
\end{cases}
\]

\[
U_2(x, y) = \sum_{n=1}^{\infty} \frac{2 (-1)^n}{n \pi \left( \frac{1}{3} n^2 +1 \right)} \sin n \pi x \cos m \pi y
\]

\[
U(x, y) = U_1(x, y) + U_2(x, y) = \frac{1}{\cosh \pi (y-1)} \sinh \pi x + \sum_{n=1}^{\infty} \frac{2 (-1)^n}{n \pi \left( \frac{1}{3} n^2 +1 \right)} \sin n \pi x \cos m \pi y
\]