Math. 6431, Final Exam.
The final exam is due on Monday, December 7 at 5:00 pm. Please either bring the papers to my office or put them in my mailbox in Skiles 117 by this deadline. In addition to the material presented in class you can also use the results from the homework problems. You cannot collaborate with other people.

1. (7 pts) Solve the initial boundary value problem

\begin{align*}
\frac{u_t}{(u^2)}_x &= 0, \quad x > 0, t > 0 \\
u(0,x) &= \frac{1}{2}, \quad x > 0 \\
u(t,0) &= \begin{cases} 
1 & 0 < t < 1 \\
\frac{1}{2} & t > 1
\end{cases}
\end{align*}

Draw a picture depicting what happens for all times \( t \geq 0 \) that include characteristics and the shock curves.

2. (7 pts) Let \( \Omega = \{ x \in \mathbb{R}^2 : |x| > 1 \} \), \( f \in C(\Omega) \), \( g \in C(\partial \Omega) \). Show that there is at most one bounded solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) of the boundary value problem

\begin{align*}
\Delta u &= f, \quad \text{in } \Omega \\
u &= g \quad \text{on } \partial \Omega.
\end{align*}

3. (7 pts) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( Q = (0, +\infty) \times \Omega \), and let \( u \in C^{1,2}(Q) \cap C(\overline{Q}) \) satisfy

\begin{align*}
\frac{u_t}{-\Delta u + c(t,x)u} &= 0, \quad ((t,x) \in Q) \\
u(t,x) &= 0, \quad (t > 0, x \in \partial \Omega) \\
u(0,x) &= g(x) \quad (x \in \Omega),
\end{align*}

where \( c(t,x) \geq \gamma > 0 \) and is continuous and \( g \) is bounded and continuous. Show that there is a constant \( C \) such that

\[ |u(t,x)| \leq Ce^{-\gamma t} \quad \text{in } Q. \]

4. (7 pts) Let \( \Omega \) be a bounded open set with a smooth boundary \( \partial \Omega \). Let \( u \in C^2([0,T] \times \Omega) \) solve

\begin{align*}
\frac{u_t}{-\Delta u + au_t + bu} &= 0, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} + \alpha u &= 0, \quad x \in \partial \Omega, t > 0,
\end{align*}

where \( a, b \) are nonnegative constants, \( \alpha = \alpha(x) \) is a continuous nonnegative function on \( \partial \Omega \), and \( \nu \) is the outward unit normal vector on \( \partial \Omega \). Prove that the energy function

\[ e(t) = \frac{1}{2} \int_{\Omega} \left( u_t^2(t,x) + |Du(t,x)|^2 + bu^2(t,x) \right) dx + \frac{1}{2} \int_{\partial \Omega} \alpha(x)u^2(t,x) dS(x) \]

is nonincreasing.

5. (6 pts) Let \( f \in C(\mathbb{R}) \) be periodic with period 1, i.e. \( f(x + 1) = f(x) \) for all \( x \in \mathbb{R} \). Show that the solution of

\begin{align*}
\frac{u_t}{-u_{xx}} &= 0, \quad (t,x) \in (0, +\infty) \times \mathbb{R}, \\
u(0,x) &= f(x), \quad x \in \mathbb{R}
\end{align*}
satisfies \( u(t, x + 1) = u(t, x) \) for all \( x \in \mathbb{R} \).

6. (6 pts) Consider the Cauchy problem

\[
\begin{aligned}
    u_t &= \cos(u_x), \quad x \in \mathbb{R}, t > 0 \\
    u(0, x) &= \frac{\pi}{4}x + \frac{\pi}{6}x^2, \quad x \in \mathbb{R}.
\end{aligned}
\]

Verify that the assumptions of the Cauchy-Kovalevskaya theorem are satisfied and then find the terms of order \( \leq 3 \) of the Taylor series of the solution about the origin.