1) There is an edge between every pair of vertices. How many pairs are there? \( \binom{4}{2} \) \( \text{Ans.} \)

2) Let's count subgraphs by the # of vertices they have.

<table>
<thead>
<tr>
<th># vertices in subgraph</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td># ways to pick that many vertices from ( K_4 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td># pairs of these vertices</td>
<td>0</td>
<td>( \frac{4}{2} )</td>
<td>( \frac{6}{2} )</td>
<td>( \frac{4}{2} )</td>
<td>( \frac{4}{2} )</td>
</tr>
<tr>
<td># choices for edges (for each pair, include an edge or don't)</td>
<td>( 2^0 )</td>
<td>( 2^0 )</td>
<td>( 2^1 )</td>
<td>( 2^1 )</td>
<td>( 2^2 )</td>
</tr>
</tbody>
</table>

Total subgraphs: \( 2^0 \cdot \left( \frac{4}{2} \right) + 2^0 \cdot \left( \frac{4}{4} \right) + 2^1 \cdot \left( \frac{4}{2} \right) + 2^1 \cdot \left( \frac{4}{2} \right) + 2^2 \cdot \left( \frac{4}{4} \right) \)

For those of you actually into numbers and stuff, that's

\[
1 + 4 + 2 \cdot 6 + 2 \cdot 4 + 2^6
\]

\[
= 1 + 4 + 12 + 32 + 64 = 113 \text{ subgraphs} \quad \text{Ans.}
\]

3) Let's list them.

0-vx

\[ \emptyset \]

1-vx

\[ \cdot \]

2-vx

\[ : \cdot \cdot \]

3-vx

\[ : \cdot \cdot \cdot \]

(No more because these can be related to cover all possibilities)

Thus, there are 8 non-isomorphic subgraphs of \( K_3 \). \( \text{Ans.} \)

\( \text{Flip=} \)
4) For any graph \( G = (V, E) \),
\[ \sum_{v \in V} \deg(v) = 2|E|. \]

**Proof:**
Consider each edge \( xy \in E \).
Since the degree of a vertex is the number of edges connected to the vertex, 
edge \( xy \) contributes 1 to the degree of \( x \) & 1 to the degree of \( y \).
Similarly, all edges contribute 1 to exactly two vertices' degrees.
Thus,
\[ \sum_{v \in V} \deg(v) = \sum_{e \in E} 2 = 2|E|! \]

5.) Exercise 5.9 #1:

a) \( \deg(8) = 2 \)

b) \( \deg(10) = 4 \)

c) The vertices of degree 2 are \{ 1, 4, 6, 8, 9 \}.
There are 5.

d) \((5, 1, 7, 4, 10, 3, 2, 6)\) among many other choices.

e) \((3, 10, 4)\) - a path of length 3. This is definitely the shortest since 3w4.

f) \((8, 3, 2, 7)\) or \((8, 9, 10, 7)\)

It's not too bad to check no path of length 1 or 2 exists.

g) \((4, 7, 10, 3, 2, 6)\) or \((4, 7, 1, 5, 2, 6)\)...

Next Pg:)}
6) Exercise 5.9 #3
No such graph exists, by the corollary to $\sum_{v \in V} \text{deg} v = 2|E|$: every graph must have an even # of vs of odd degree, but the proposed graph has 3 such vs, with degrees 5, 1, & 1.

7) Exercise 5.9 #6
Every tree on $n$ vertices has $n-1$ edges.
Proof: By induction.
Base case: $n=1$. The only tree is $\cdot$, which has 0=1-1 edges.
Inductive step: Assume every tree with $k$ vs, $1 \leq k \leq m$ (for a fixed $m \in \mathbb{N}$), has $k-1$ edges.
Take a tree $T$ with $m+1$ vs.
Remove an edge - $T$ is divided into 2 subtrees $T_1$ & $T_2$.

$T_1$ & $T_2$ both have a # vs between 1 and $m$, so the inductive hypothesis holds. So,

$\rightarrow T_1$ has $k_1$ vs & $k_1-1$ edges
$\rightarrow T_2$ has $k_2$ vs & $k_2-1$ edges
$\rightarrow k_1+k_2 = m+1$ because $T_1$ & $T_2$ form $T$ when the edges are added back.

Thus, the number of edges in $T$ is

\[ (k_1-1) + (k_2-1) + 1 = k_1 + k_2 - 1 = (m+1) - 1 \text{ edges} \]