This exam contains 8 questions worth 10 points each, for a total of 80 points. Not anymore!

The exam is closed-book. You may not use any outside sources.

Score Table (for instructor use only)

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I have adhered to the Georgia Tech Honor Code while completing this exam.

(Sign here)
1. (10 points)

(a) (5 points) What is the coefficient of \(x^2y^5\) in the expansion of \((x + 2y + 3)^{17}\)?

By the Multinomial Theorem, it's

\[25 \cdot 3^{10} \cdot \binom{17}{2, 5, 10}\]

Ans.

(b) (5 points) How many ways can you rearrange the letters in the phrase, IHATETESTS?

This phrase has

- I - 1
- H - 1
- A - 1
- T - 3
- E - 2
- S - 2

So, it's

\[
\binom{10}{1, 1, 1, 3, 2, 2} = \frac{10!}{3!2!2!}
\]

Ans.
2. (10 points) Show that if nine points are put anywhere within an equilateral triangle with side length two, then there are always at least three points within distance at most one of each other. (In other words, every pair from these three points is within distance one of each other.)

We can divide the Δ into 4 equilateral Δ's of side length 1. By the PHP, since
\[ 9 = 2 \cdot 4 + 1, \]
no matter how the dots are placed, at least
1 Δ has at least 3 dots.

Since the Δ has side length 1, these dots are within 1 of each other.
3. (10 points) How many lattice paths are there from \((0, 0)\) to \((5, 7)\) which use edges only from the following diagram?

Possible paths:

A. \((0, 0) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (5, 7)\)

B. \((0, 0) \rightarrow (2, 3) \rightarrow (3, 3) \rightarrow (5, 7)\)

C. \((0, 0) \rightarrow (2, 5) \rightarrow (3, 5) \rightarrow (5, 7)\)

These are the only ways to get from \((0, 0)\) to \((5, 7)\), and they don't overlap. So, the total # of paths is

\[
\begin{align*}
\binom{3}{2} \cdot \binom{8}{2} + \binom{5}{2} \cdot \binom{4}{4} + \binom{7}{2} \cdot \binom{4}{2} \\
A 18,
\end{align*}
\]
(a) (5 points) As in class, let \( N = \{1, 2, 3, \ldots \} \). Is \(|N| = |\mathbb{Z}_{\geq 0}|\)? Prove your answer.

**Yes.**

Let \( f(n) = n - 1 \). \( f \) is a bijection since it hits each \( z \in \mathbb{Z}_{\geq 0} \) exactly once:

\[
\begin{array}{cccccc}
1, & 2, & 3, & 4, & 5, & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0, & 1, & 2, & 3, & 4, & \ldots \\
\end{array}
\]

(b) (5 points) Let \((0, 1)\) be the set of real numbers between 0 and 1, excluding 0 and 1. Consider the map \( f : N \to (0, 1) \) defined as follows: for any number \( n = n_1n_2 \ldots n_k \), \( f(n) = 0.n_kn_{k-1} \ldots n_1 \). For example, \( f(12) = 0.21 \) and \( f(399) = 0.993 \). Is \( f \) a bijection?

**No.**

\( f \) is NOT onto, because it misses any decimals with infinitely many digits, such as 0.1111...
5. (10 points) As in class, let $P(n,k)$ be the number of permutations of length $k$ from an $n$-letter alphabet. Give a combinatorial proof of the following:

$$P(n,k) = P(n-1,k) + k \cdot P(n-1,k-1)$$

LHS = \# permutations of length $k$ from $[n]$.

RHS:

Case 1: The permutation \underline{doesn't} use "n."

Then, we make a permutation from $[n-1]$ of length $k$. There are $P(n-1,k)$ choices.

Case 2: The permutation \underline{does} use "n."

There are $P(n-1,k-1)$ ways to find the rest of the permutation, since it's of length $k-1$ from $[n-1]$. Then, there are $k$ spots where we can put in $n$:

\[\begin{array}{c}
\text{k spots to put "n"}
\end{array}\]

So, there are $k \cdot P(n-1,k-1)$ such permutations.

Since these are the only 2 cases for permutations of length $k$,

$$P(n,k) = P(n-1,k) + k \cdot P(n-1,k-1) \checkmark$$
6. (10 points) Let \( r(1) = 1 \), and \( r(n) = r(n-1) + 2(n-1) \) for \( n \geq 2 \). Show that \( r(n) = O(n^2) \).

**Claim:** \( r(n) \leq n^2 \) for all \( n \).

**Proof:**

**Base case:** For \( n = 1 \),

\[
\begin{align*}
r(1) &= 1 \\
1^2 &= 1
\end{align*}
\]

Thus, \( r(1) \leq 1^2 \checkmark \)

**Inductive Step:** Assume \( r(k) \leq k^2 \) for some \( k \in \mathbb{N} \).

Then,

\[
\begin{align*}
r(k+1) &= r(k) + 2k \\
&\leq k^2 + 2k \\
&\leq k^2 + 2k + 1 \\
&= (k+1)^2 \checkmark
\end{align*}
\]

Thus, \( r(n) = O(n^2) \).
7. (10 points) Let $g(n)$ be the number of ways of coloring a $1 \times n$ checkerboard with black or white squares, where no three consecutive squares are white. Find a recursive equation and base cases for $g(n)$. (Consider horizony.

See next page for alternate solution.

Let’s see how the board can end:

\[
\begin{array}{c}
 n-1 \\
 B \\
 n-2 \\
 B \\
 W \\
\end{array}
\]

Any good board

Any good board

These are the ONLY three cases. Thus,

\[g(n) = g(n-1) + g(n-2) + g(n-3)\]

We need $g(1)$, $g(2)$, & $g(3)$

\[
\begin{aligned}
g(1) &= 2 \text{ (any board length 1)} \\
g(2) &= 4 \text{ (any board length 2)} \\
g(3) &= 8 - 1 = 7 \text{ (any board length 3 - WWW)}
\end{aligned}
\]
7) Alternate solution.

Consider the last square on the checkerboard. There are only 2 cases:

Case 1:
If the last square is B, any \( 1 \times (n-1) \) checkerboard can be appended to the front to get a \( 1 \times n \) "good" checkerboard. → There are \( g(n-1) \) such choices.

Case 2:
If the last square is W, any "good" \( 1 \times (n-1) \) checkerboard can be appended, except those ending in WW. How many are there?

\[
\begin{array}{c|c|c}
\text{spots} & \text{WW} \\
\hline
\end{array}
\]

→ MUST be black for the board to be "good"

Since the third-to-last square must be black, and then any "good" board of length \( n-4 \) can be appended to get a "good" board of length \( n-1 \) ending in WW, there are \( g(n-4) \) such choices.

→ Thus, in total, there are

\[
g(n-1) - g(n-4) \quad \text{boards of length } n \text{ ending in W}.
\]

Summing over the two cases yields

\[
g(n) = 2g(n-1) - g(n-4) \quad \text{Ans.}
\]

We must now also give \( g(4) \).
Any \( 2 \times 4 \) checkerboard is allowed except 3:

\[
\begin{array}{cccc}
\text{WWW} & \text{B} \\
\text{B} & \text{WWW} \\
\text{WWW} & \text{WWW}
\end{array}
\]

Thus, \( g(4) = 2^4 - 3 = 13 \quad \text{Ans.} \)
How many solutions are there to the following?

\[ x_1 + x_2 + x_3 + x_4 < 30, \text{ with } x_1, x_2 > 0, x_3 \geq 13, \text{ and } x_4 \leq 5 \]

Equivalently,

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 30, \]

\[ x_1, x_2 > 0, x_3 \geq 13, x_4 \leq 5, x_5 > 0 \]

or, preassigning 12 to \( x_3 \),

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 18 \]

\[ x_1, x_2, x_3, x_5 \geq 0, x_4 \leq 5 \]

Related problems:

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 18 \]

\[ x_1, x_2, x_3, x_5 \geq 0, x_4 > 5 \]

Equivalent \( \rightarrow \)

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 13 \]

\[ x_1, x_2, \ldots, x_5 \geq 0 \]

\( \binom{12}{4} \)

Subtract the two:

\[ \binom{19}{4} - \binom{12}{4} \]

Answer: 18