This exam contains 7 questions worth 10 points each, for a total of 70 points.

The exam is closed-book. You may not use any outside sources.

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I have adhered to the Georgia Tech Honor Code while completing this exam.

(Sign here)
1. (10 points)

(a) (5 points) Prove or disprove: For all graphs with the same number of vertices and edges, the number of vertices of odd degree is also the same.

False.

(b) (5 points) Can you remove a single edge from $K_6$ to make it planar? Prove your answer.

No.

$K_6$ has $\binom{6}{2} = \frac{6\cdot5}{2} = 15$ edges

So, without an edge, it still has 14 edges.

But, $3n-5 = 3\cdot6-5 = 13$

So, $m > 3n-5$ & thus $K_6$-edge is still not planar.

Alternatively -- it still contains a $K_5$. 
2. (10 points) Consider the following graph:

(a) (5 points) Is the graph planar? Prove your answer.

(b) (5 points) What is the chromatic number of the graph? Prove your answer.

\[ \chi(G) = 3. \]

\[ \chi(G) = 3 \text{ b/c } d, f, g \text{ is a 3-clique.} \]

\[ \chi(G) \leq 3 \text{ because I found a 3-coloring above.} \]
3. (10 points) How many subgraphs does the following graph have? You do not need to simplify your answer. Make sure your counting strategy is clear.

Let's count the subgraphs of $K_4$:

<table>
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<tr>
<th>0 vs 5</th>
<th>1 vs 5</th>
<th>2 vs 5</th>
<th>3 vs 5</th>
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<tr>
<td>1 graph</td>
<td>1 graph</td>
<td>(4) graphs</td>
<td>(4) ways of picking vs 5</td>
<td>(4) ways of picking vs 5</td>
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<tr>
<td>1 possible edge to include</td>
<td>2 (4) subgraphs</td>
<td>1 way of picking vs 5</td>
<td>2 (4) possible edges</td>
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<td>2 vs 5</td>
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<td>3 vs 5</td>
<td>(4) ways of picking vs 5</td>
<td>(4) ways of picking vs 5</td>
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<td>4 vs 5</td>
<td>(4) ways of picking vs 5</td>
<td>(4) ways of picking vs 5</td>
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Now, for each subgraph, we can choose whether or not to include $\varepsilon_1 \varepsilon_3$. That gives

$$Q = \left(1 + \binom{4}{1} + 2 \cdot \binom{4}{2} + \binom{4}{3} \right) \cdot \binom{3}{2} + \binom{4}{2} \cdot \binom{4}{2}$$
4. (10 points) Which pairs of graphs are isomorphic? Justify your answers by defining an isomorphism or stating why one cannot exist.

(a) (3 points)

Yes.

\[ \phi(1) = 1 \]
\[ \phi(2) = 4 \]
\[ \phi(3) = 3 \]
\[ \phi(4) = 2 \]

(b) (3 points)

No. \( \deg(4) = 3 \) on left, but all vxs are of degree 2 on the right. \( \Delta \) isomorphisms preserve degree.

(c) (4 points)

Yes.

\[ \phi(1) = 3 \]
\[ \phi(2) = 4 \]
\[ \phi(3) = 2 \]
\[ \phi(4) = 8 \]
\[ \phi(5) = 1 \]
\[ \phi(6) = 7 \]
\[ \phi(7) = 5 \]

\text{cycle: } \phi(8) = 6

\text{path: } \phi(7) = 5
5. (10 points)

(a) (5 points) Consider a poset \( P = (X, \leq) \), and take any subset \( A \subseteq X \). Is \( A \) always a chain or an antichain? Prove your answer.

\[
\text{Nop. In } B_3, \\
\{ \emptyset, \{1,3\}, \{2,3\} \} \text{ is neither:} \\
\emptyset \leq \{1,3\} \text{ so this pair is comparable} \\
\{1,3\} \& \{2,3\} \text{ aren't comparable.}
\]

(b) (5 points) Is the following graph Hamiltonian? Justify your answer.

\[
\text{Nop. The circled vertex is a choke point which would need to be hit twice in a cycle around all the other vertices (or 3 times if it's the start/end point), which isn't allowed in a cycle.}
\]
6. (10 points) Prove that a graph with at most two odd-degree vertices has an Eulerian trail.

Start with an arbitrary graph $G$ with 2 odd-degree vertices $x$ and $y$.

$\rightarrow$ Add an edge between $x$ and $y$.

(If $x$ and $y$ were connected, we now have a multigraph - that's OK!)

$\rightarrow$ Now, all vertices have even degree.

$\rightarrow$ By a theorem, this graph therefore MUST have an Eulerian circuit.

$\rightarrow$ An Eulerian circuit uses each edge exactly once. Remove the edge $xy$ from the circuit, & we're left with a trail in the original graph $G$.

That is, if the circuit was

$$(S_1, S_2, \ldots, S_m, x, y, S_{m+3}, \ldots, S_n)$$

with $S_1 = S_n$,

the trail is

$$(y, S_{m+3}, S_{m+4}, \ldots, S_n, S_2, S_3, \ldots, S_m)$$

(If the circuit had "y,x" instead, we could cut it similarly.)

Thus, $G$ has an Eulerian trail. $\square$
7. (10 points)

(a) (5 points) Consider \( P = (X, \leq) \) with \( X = \{1, 2, 3, 4, 5, 6\} \) and \( R \) defined as follows:

\[
R = \{(1, 1), (2, 2), \ldots, (6, 6), (2, 4), (2, 6), (4, 6), (6, 5), (4, 3), (2, 1), (2, 3),
(2, 5), (4, 1), (4, 5), (3, 5), (1, 5), (3, 2)\}
\]

If \( P \) is a poset, draw its Hasse diagram. If it is not, determine why not.

Let's list the properties:

1 \( \leq \) 5
2 \( \leq \) 4, 6, 1, 3, 5
3 \( \leq \) 5, 2
4 \( \leq \) 6, 3, 1, 5
5 \( \leq \) \( \emptyset \)
6 \( \leq \) 5

\[ \Rightarrow 3 \leq 2, 2 \leq 3 \quad \# \]

NOT antisymmetric
(b) (5 points) Consider $P = (X, \leq)$ with $X = \{1, 2, 3, 4, 5, 6\}$ and $R$ defined as follows:

$$R = \{(1, 1), (2, 2), \ldots, (6, 6), (2, 3), (5, 6), (5, 3), (1, 3), (2, 6), (4, 3), (3, 6), (1, 6), (4, 6)\}$$

If $P$ is a poset, draw its Hasse diagram. If it is not, determine why not.

Let's list 'em again:

$1 \leq 3, 6$

$2 \leq 3, 6$

$3 \leq 6$

$4 \leq 3, 6$

$5 \leq 6, 3$

$6 \leq 3$