This exam contains 7 questions worth 10 points each, for a total of 70 points.

The exam is closed-book. You may not use any outside sources.

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td><strong>70</strong></td>
<td></td>
</tr>
</tbody>
</table>

I have adhered to the Georgia Tech Honor Code while completing this exam.

[Signature]
1. (10 points)

(a) (3 points) Explain why an Eulerian circuit cannot exist in a graph with at least one odd-degree vertex.

In an Eulerian circuit, the path must enter and leave a vertex the same number of times in order to end up where it started. Since you can't reuse edges, each vertex is attached to an even number of edges.

(b) (7 points) Find an Eulerian circuit in the following graph, or prove one doesn't exist:

Partial circuit: $(1, 3, 2, 4, 5, 7)$
Partial circuit: $(2, 5, 6, 2)$ combine: $(1, 3, 2, 5, 6, 2, 4, 5, 1)$
Partial circuit: $(6, 4, 7, 6)$
Total: $(1, 3, 2, 5, 6, 4, 7, 6, 2, 4, 5, 1)$
2. (10 points)

(a) (3 points) A graph has vertices with degrees 6, 4, 4, 3, 2, 1, 1, 1. Can we conclude how many edges are in the graph? If so, how many are there?

\[ \sum \text{deg}(v) = 2|E|, \text{ so,} \]

\[ |E| = \frac{6+4+4+3+2+1+1+1}{2} = \frac{18}{2} = 9 \text{ edges} \]

(b) (7 points) How many subgraphs does the following graph have?

Subgraphs of \(G\):
\[ \emptyset, 1, 2, 3, 4, 5 \]

Subgraphs of \(H\): 5 of 'em.

\[ 0 \times 0 + 1 \times 0 + 2 \times 0 + 3 \times 0 = 0 \]

\[ 1 + 3 + 6 + 8 = 18 \]

Thus, the total is \[ 18 + 5 = 23 \text{ subgraphs} \]
3. (10 points) Consider the boolean lattice, $B_6$.

(a) (3 points) What is the largest antichain in $B_6$, and what is its size? Justify your answer. (A clear verbal description of the antichain is sufficient.)

By Sperner’s Theorem, the largest antichain is the subsets of $[6]$ of size 3. There are

\[
\binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ such subsets.}
\]

(b) (3 points) Consider the set $\{1, 5, 6\} \subset [6]$. How many elements $A \subseteq [n]$ are there in $B_6$ with the property that $A \geq \{1, 5, 6\}$?

We have three options to add $2, 3, \& 4$ to the set $\{1, 5, 6\}$, giving

\[
2^3 = 8 \text{ sets } \geq \{1, 5, 6\} \text{ Ans.}
\]

(c) (4 points) How many maximal chains are there from $\{1, 5, 6\}$ to $\{1, 2, 3, 4, 5, 6\}$? (That is, how many chains are there which have $\{1, 5, 6\}$ as their smallest element, $\{1, 2, 3, 4, 5, 6\}$ as their largest element, and are as long as possible?)

\[311\text{ Ans.}\]

We can choose the order in which we add $2, 3, \& 4$ to $\{1, 5, 6\}$ to get from $\{1, 5, 6\}$ to $\{1, 2, 3, 4, 5, 6\}$.
4. (10 points) Use induction to show that every tree is 2-colorable. Do not use the fact that a graph is 2-colorable if and only if it has no odd cycles.

**Proof:** by induction.

**Base case:** \( n=1 \). The only tree is \( T_1 = *1 \).

It is 1-colorable, & thus 2-colorable.

**Inductive step:** Assume all trees with \( k \) vs are 2-colorable.

Let \( T \) be a tree w/ \( k+1 \) vs.

\( T \) has at least 2 leaves.

Remove a leaf \& an edge.

By inductive hypothesis, the remaining tree is 2-colorable.

Add back in the leaf, and color it the opposite color of its parent. This is a proper 2-coloring.

Thus, \( T \) is 2-colorable.
5. (10 points) Consider the complete graphs $K_n$.
(a) (3 points) For which $n \geq 3$ is $K_n$ planar? Justify your answer.

A graph is planar if it contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$.

Thus, they aren't planar.

(b) (3 points) For which $n \geq 3$ is $K_n$ Eulerian? Justify your answer.

Every vertex must be of even degree.

Each vertex in $K_n$ has $n-1$ nbs, so we need $n-1$ to be even $\Rightarrow$ for all $n$ odd, $n \geq 3$.

(c) (4 points) For which $n \geq 2$ is $K_n$ Hamiltonian? Justify your answer.

The "outer edge" of $K_n$ is a cycle $\Rightarrow (1,2,3,\ldots, n-1,n)$ forms a Hamiltonian cycle.

Thus, $K_n$ is Hamiltonian for all $n \geq 3$. 
6. (10 points) The following questions refer to the poset $P$ whose Hasse diagram is drawn below.

(a) (3 points) What is the height of $P$? Justify your answer.

\[ \{h, g, f, a, 3\} \text{ is a maximum-length chain, so} \]

\[ \text{height}(P) = 4. \]

(b) (2 points) Partition $P$ into $\text{height}(P)$ antichains.

\[ \{h, i, d, c, 3\}, \{g, 3\}, \{e, f\}, \{e, a, b, c, 3\} \]

(c) (3 points) What is the width of $P$? Justify your answer.

\[ 4 \text{ is a maximum-length antichain is } \{h, i, d, c, 3\}. \]

(d) (2 points) Partition $P$ into $\text{width}(P)$ chains.

\[ \{h, g, f, a, 3\}, \{e, c, 3\}, \{d, b, 3\}, \{i, 3\} \]
7. (10 points) Radio stations broadcasting on the same frequency interfere with each other whenever they are within 90 kilometers of each other. There are six radio stations in a city. The distance between them is given in the matrix below, where the \((i,j)\)th entry is the distance in kilometers between stations \(i\) and \(j\). What is the least number of frequencies needed to assign each radio station a frequency without having stations interfere with each other? Prove your answer.

We can convert this to a graph-coloring problem. Stations interfere if there's an edge between them.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 42 & 60 & 85 & 95 & 60 \\
2 & 42 & 0 & 42 & 95 & 120 & 95 \\
3 & 60 & 42 & 0 & 60 & 95 & 85 \\
4 & 85 & 95 & 60 & 0 & 42 & 60 \\
5 & 95 & 120 & 95 & 42 & 0 & 42 \\
6 & 60 & 95 & 85 & 60 & 42 & 0
\end{pmatrix}
\]

\[
\begin{array}{c}
\text{\{1, 3, 4, 6\} is a 4-clique, so the chromatic number is at least 4.}
\end{array}
\]

We can indeed 4-color the graph.

Thus, we need 4 frequencies.