Homework 7
Math 3012-N
Due: Friday, October 23, in class or in office hours

Please submit hard copies only. Does anyone read this? Staple and remove fringe.

1. Let $P = (\mathbb{Z}, \leq^*)$, with $\leq^*$ defined as follows: for $a, b \in \mathbb{Z}$, $a \leq^* b$ if and only if $|a - 1| \leq |b - 1|$ as integers. Is $P$ a poset? Either prove that it is, or describe why it is not.

2. Do Exercise 6.10 #1. Here, a relation $R$ is just a subset of $X \times X$ – that is, $R$ is a collection of ordered pairs from $X$. This problem does not require that the relation be transitive, antisymmetric, or reflexive.

3. Do Exercise 6.10 #3.


5. Remember the following definitions:
   - Two posets $P = (X, \leq_1)$ and $Q = (Y, \leq_2)$ are isomorphic if there is a bijection $f : X \to Y$ such that for any $a, b \in X$, $a \leq_1 b$ if and only if $f(a) \leq_2 f(b)$.
   - The cover graph of a poset $P = (X, \leq_1)$ is the graph with vertex set $X$ and edges between $a$ and $b$ if and only if $a$ covers $b$ or $b$ covers $a$.
   - The Hasse diagram of a poset $P = (X, \leq_1)$ is the cover graph with the extra restriction that if $a$ covers $b$, $a$ is vertically above $b$ in the graph.

   (a) Posets $P = (X, \leq_1)$ and $Q = (X, \leq_2)$ have the same Hasse diagram. Must they be isomorphic? (Note that the ground sets are the same here.) Justify your answer.

   (b) Posets $P = (X, \leq_1)$ and $Q = (X, \leq_2)$ have the same cover graph. Must they be isomorphic? Justify your answer.

6. Let $P$ be a poset with width $w$. Prove that there is no partition of $P$ into $w - 1$ chains.

7. For a poset $P = (X, \leq)$, we can define a dual poset $P^* = (X, \leq^*)$ as follows: $a \leq^* b$ in $P^*$ if and only if $b \leq a$ in $P$.

   (a) Is it always true that $P$ and $P^*$ have the same cover graph? Prove or disprove.

   (b) Prove that if $P$ is a total order (and $X$ is finite), then $P$ is isomorphic to $P^*$.

   (c) Is the following statement true? The only posets $P$ where $P$ is isomorphic to $P^*$ are total orders. (Consider the Hasse diagrams of a poset and its dual.)

8. Let $P = ([a, b, c, d, e, f, g, h, i, j], R)$ with $R$ defined as follows:

   $$R = \{(a, a), (b, b), \ldots, (j, j), (f, a), (f, e), (d, h), (g, f), (i, j), (g, e), (b, e), (c, h), (c, b), (c, d), (c, e), (d, e), (g, a)\}$$

   Draw the Hasse diagram for $P$.

9. Define the poset $P_n = (T_n, \leq)$ as follows:
   - $T_n$ is the set of ternary strings of length $n$.
   - For two ternary strings $s = s_1s_2\ldots s_n, t = t_1t_2\ldots t_n$, we have that $s \leq t$ in $P_n$ if and only if each digit $s_i \leq t_i$, as integers, for all $i \in [n]$.

   For example, $001 \leq 021 \leq 122$, but $010$ and $002$ are not comparable.
(a) Draw the Hasse diagram for $P_2$.
(b) What is the width of $P_2$? The height?
(c) Previously, we saw that binary strings with the same relation were isomorphic to the boolean lattice, which is defined in terms of subsets of $[n]$. How can we generalize the notion of “subsets of $[n]$” to account for the new ternary lattice?