INTRODUCTION TO PLANAR GRAPHS

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ABSTRACT

The topic of planar graphs is covered in many books and articles, but the treatment usually relies on intuition or on deep topological theorems that are quoted without proof. I give a self-contained rigorous introduction to planar graphs.

1 The Jordan curve theorem and Euler’s formula

Throughout these notes graphs are allowed to have loops and multiple edges.

Definition 1.1. A polygonal arc is a set $A \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the interval $[0,1]$. The images of 0 and 1 under the homeomorphism are called the ends of $A$. A polygon is a set $B \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. By a bend of a polygonal arc or a polygon $P$ we mean a point of $P$ where two different straight line segments meet. Thus $P$ has finitely many bends.

Definition 1.2. Let $\Omega \subseteq \mathbb{R}^2$ be an open set, and for $x, y \in \Omega$ let us define $x \sim y$ if there exists a polygonal arc $A \subseteq \Omega$ with ends $x$ and $y$. Then $\sim$ is an equivalence relation, and the equivalence classes are called the arcwise connected components of $\Omega$. If $x \sim y$ for any two $x, y \in \Omega$, then we say that $\Omega$ is arcwise connected. Now if $F \subseteq \mathbb{R}^2$ is closed, we say that an arcwise connected component of $\mathbb{R}^2 - F$ is a face of $F$.

Theorem 1.3 (Jordan curve theorem for polygons). Every polygon $P$ has exactly two faces, of which exactly one is bounded. The boundary of each of the two faces is $P$.

Proof. For $x \in \mathbb{R}^2 - P$ and a half-line $L$ originating in $x$ and containing no bends of $P$ let $\pi(x,L)$ denote the number of intersections of $L$ with $P$ modulo 2. It is easy to check that this can be extended to all half-lines originating in $x$ in such a way that $\pi(x,L)$ does not depend on $L$. Let us call that value $\pi(x)$. Furthermore, it follows that the function $\pi$ is continuous, and hence is constant on each arcwise connected component of $\mathbb{R}^2 - P$. By choosing two points $x_1$ and $x_2$ close

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to each other, but on opposite sides of a straight-line segment of \( P \) we can show that \( \pi(x_1) \neq \pi(x_2) \). Thus \( P \) has at least two faces.

Suppose now that \( P \) has at least three faces, and choose a point in each, say \( x_1, x_2, x_3 \). Pick a point \( x \) on the boundary of \( P \) inside a straight-line segment \( S \) of \( P \). Pick a small open neighborhood \( O \) of \( x \) such that \( O \cap P = O \cap S \). By shooting a half-line from each of \( x_1, x_2, x_3 \) toward \( P \) and then following the boundary of \( P \) until we hit \( O \) we see that each of \( x_1, x_2, x_3 \) can be reached from a point in \( O \) by a polygonal arc not intersecting \( P \). But \( O - P \) has at most two arcwise connected components, and hence \( P \) has at most two faces. It follows that every point of \( O \cap S \) belongs to the boundary of both faces of \( P \), and since \( x \) was arbitrary, we deduce that the boundary of both faces of \( P \) is \( P \). We leave it as an exercise to show that exactly one of the faces is bounded. \( \square \)

**Definition 1.4.** A plane graph is graph \( G \) such that
(i) \( V(G) \subseteq \mathbb{R}^2 \),
(ii) for every non-loop edge \( e \in E(G) \) with ends \( u \) and \( v \) there exists a polygonal arc \( A \) with ends \( u \) and \( v \) such that \( e = A - \{u, v\} \subseteq \mathbb{R}^2 - V(G) \),
(iii) for every loop \( e \in E(G) \) incident with \( u \in V(G) \) there exists a polygon \( P \) containing \( u \) such that \( e = P - \{u\} \subseteq \mathbb{R}^2 - V(G) \), and
(iv) if \( e, e' \in E(G) \) are distinct, then \( e \cap e' = \emptyset \).

Thus an edge of a plane graph is a subset of the plane that includes no vertices, not even its ends. We define the point set of \( G \) to be the set \( \bigcup_{e \in E(G)} e \cup V(G) \), and by abusing notation we shall denote this set also by \( G \). Thus the faces of \( G \) are the arcwise connected components of \( \mathbb{R}^2 - G \). The set of faces of a plane graph \( G \) will be denoted by \( F(G) \).

**Definition 1.5.** A graph \( G \) is planar if it is isomorphic to a plane graph \( \Gamma \). We say that \( \Gamma \) is a (planar) drawing of \( G \).

**Lemma 1.6.** Let \( G \) be a plane graph, let \( e \in E(G) \), and let \( x_1, x_2 \in \mathbb{R}^2 - G \) be two points such that the straight line segment connecting them intersects \( e \) exactly once, and is otherwise disjoint from \( G \). Let \( f_1 \) be the face of \( G \) that includes \( x_1 \). Then \( e \) is a subset of the boundary of both \( f_1 \) and \( f_2 \) and is disjoint from the boundary of every other face of \( G \).

**Proof.** This follows by a similar argument as the second half of the proof of Theorem 1.3. \( \square \)

**Corollary 1.7.** If \( G \) is a plane graph and \( f \in F(G) \), then the boundary of \( f \) is the point set of a subgraph of \( G \). In particular, if the boundary of \( f \) includes a point belonging to an edge \( e \in E(G) \), then it includes the entire edge \( e \).

**Definition 1.8.** Let \( G, e, f_1, f_2 \) be as in Lemma 1.6. We will refer to \( f_1 \) and \( f_2 \) as the two faces incident with \( e \). Please note that \( f_1 \) and \( f_2 \) need not be distinct.

**Lemma 1.9.** Let \( G \) be a plane graph that is a forest. Then \( G \) has exactly one face.

**Proof.** Exercise. Use induction on the number of vertices plus the number of bends. \( \square \)

**Lemma 1.10.** Let \( G' \) be a subgraph of a plane graph \( G \). Then
(i) every face of \( G \) is a subset of a face of \( G' \),
(ii) if \( f \) is a face of \( G \) and \( bd(f) \subseteq G' \), then \( f \) is a face of \( G' \), and
(iii) if \( f' \in F(G') \) is disjoint from \( G \), then \( f' \in F(G) \).

**Proof.** Statement (i) follows immediately: a face \( f \) of \( G \) is an arcwise-connected subset of \( \mathbb{R}^2 - G \), and hence is a subset of an arcwise-connected component of \( \mathbb{R}^2 - G \), that is, a face of \( G' \).

To prove (ii) let \( f \) be a face of \( G \). By (i) there exists a face \( f' \) of \( G' \) such that \( f \subseteq f' \). If \( f = f' \), then (ii) holds, and so we may assume that the inclusion is proper. Hence there exists a point \( x' \in f' - f \). Let \( x \in f \), and let \( A \subseteq f' \) be a polygonal arc with ends \( x, x' \). Since \( A \) is not a subset
of \( f \), it intersects the boundary of \( f \), say in a point \( z \). But \( z \in f' \), and hence \( z \not\in G' \), contrary to hypothesis. This proves (ii).

To prove (iii) let \( f' \in F(G') \) be disjoint from \( G \). Then \( f' \) is an arcwise connected subset of \( \mathbb{R}^2 - G \), and therefore is a subset of a face \( f \) of \( G \). By (i) we have \( f \subseteq f' \), and hence \( f' = f \in F(G) \), as desired. □

**Lemma 1.11.** Let \( e \) be an edge of a plane graph \( G \) that belongs to a cycle of \( G \), and let \( f_1, f_2 \in F(G) \) be the two faces incident with \( e \). Then \( f_1 \neq f_2 \).

**Proof.** For \( i = 1, 2 \) the face \( f_i \) is a subset of a face \( f'_i \) of \( C \), by Lemma 1.10(i). By Lemma 1.6 the edge \( e \) is disjoint from the boundary of all faces of \( F(G) - \{f_1, f_2\} \), and hence it is disjoint from the boundary of all faces of \( F(C) - \{f'_1, f'_2\} \). But the edge \( e \) belongs to the boundary of both faces of \( C \) by Theorem 1.3, and hence \( f'_1 \neq f'_2 \), implying that \( f_1 \neq f_2 \), as desired. □

**Lemma 1.12.** Let \( G \) be a plane graph, let \( e \in E(G) \), and let \( f_1 \) and \( f_2 \) be the two faces incident with \( e \). Let \( f_{12} \) denote the point set \( f_1 \cup e \cup f_2 \). Then \( f_{12} \) is a face of \( G \setminus e \) and \( F(G) - \{f_1, f_2\} = F(G \setminus e) - \{f_{12}\} \).

**Proof.** To prove the first assertion we notice that \( f_{12} \) is arcwise connected, and hence is a subset of a face \( f' \) of \( G \setminus e \). We may assume that \( f_{12} \) is a proper subset of \( f' \), for otherwise \( f_{12} \in F(G \setminus e) \), as desired. Thus there exists a polygonal arc \( A \) with ends \( x \in f_{12} \) and \( y \in f' - f_{12} \) such that \( A \subseteq f' \). By considering a proper subset of \( A \) we may assume that \( A \cap e = \emptyset \). Since \( y \in f' - f_{12} \subseteq \mathbb{R}^2 - G - f_1 - f_2 \) it follows that \( y \) belongs to a face of \( G \) other than \( f_1 \) or \( f_2 \), but then \( A \) connects points belonging to different faces of \( G \), and yet \( A \cap G = \emptyset \), a contradiction. Thus \( f_{12} \in F(G \setminus e) \).

For the second assertion let \( f \in F(G) - \{f_1, f_2\} \). Since \( f_1, f_2 \) are the only two faces of \( G \) incident with \( e \) by Lemma 1.6, we deduce that \( bd(f) \subseteq G \setminus e \), and hence \( f \in F(G \setminus e) \) by Lemma 1.10(ii). Clearly \( f \neq f_{12} \). Conversely, let \( f' \in F(G \setminus e) - \{f_{12}\} \). Then \( f' \cap G = \emptyset \), and hence \( f' \in F(G) \) by Lemma 1.10(iii). Further, \( f' \notin \{f_1, f_2\} \), because \( f_1, f_2 \notin F(G \setminus e) \).

**Theorem 1.13** (Euler’s formula). Every connected plane simple graph \( G \) satisfies \( |V(G)| + |F(G)| = |E(G)| + 2 \).

**Proof.** We proceed by induction on \( |E(G)| \). If \( G \) has no cycles, then it is a tree, and the formula holds by Lemma 1.9. Thus we may assume that \( G \) has a cycle \( C \). Let \( e \in E(C) \), and let \( f_1, f_2 \in F(G) \) be the two faces incident with \( e \). Then \( f_1 \neq f_2 \) by Lemma 1.11. The formula follows from Lemma 1.12 by induction applied to the graph \( G \setminus e \).

**Corollary 1.14.** Every simple planar graph \( G \) on \( n \geq 3 \) vertices has at most \( 3n - 6 \) edges. Moreover, if \( G \) has no triangles, then it has at most \( 2n - 4 \) edges.

**Proof.** We may assume that \( G \) is connected, for an edge joining vertices in different components of \( G \) may be added without violating planarity. Let \( q \) be the number edge-face incidences \((e,f)\), with the proviso that if \( f_1 \) and \( f_2 \) are the two faces incident with \( e \) and \( f_1 = f_2 \), then the incidence \((e,f_1) = (e,f_2)\) is counted twice. Then \( q = 2|E(G)| \). On the other hand, since \( G \) has no loops or parallel edges, each face contributes at least three toward \( q \), and hence \( q \geq 3|F(G)| \). Thus \(|F(G)| \leq 2|E(G)|/3 \), and substituting this into Euler’s formula gives the first inequality. The second inequality follows similarly. □

**Corollary 1.15.** The graphs \( K_5 \) and \( K_{3,3} \) are not planar.

**Proof.** This follows immediately from Corollary 1.14. □
2 Kuratowski’s theorem

A subdivision of a graph $H$ is a graph obtained from $H$ replacing the edges of $H$ by internally disjoint paths of nonzero length with the same ends. An $H$ subdivision in a graph $G$ is a subgraph of $G$ isomorphic to a subdivision of $H$. From Corollary 1.15 we deduce

Corollary 2.1. No planar graph has a $K_5$ or $K_{3,3}$ subdivision.

The objective of this section is to prove Kuratowski’s theorem, the converse of Corollary 2.1.

Lemma 2.2. Let $G$ be a plane graph consisting of two vertices and three internally disjoint paths $P_1, P_2, P_3$ joining them. Then $G$ has precisely three faces with boundaries $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$, respectively.

Proof. The graph $P_1 \cup P_2$ has exactly two faces by Theorem 1.3; let $f_3$ be the one disjoint from $P_3$. Let $f_1, f_2$ be defined similarly. Then $f_1, f_2, f_3 \in F(G)$ by Lemma 1.10(iii), and they are distinct, because their boundaries are distinct. Conversely, let $f$ be a face of $G$, and let $x \in bd(f) - V(G)$. From the symmetry we may assume that $x$ belongs to the point set of $P_1$. Now $P_1$ is incident with at most two faces of $G$ by Lemma 1.6, and hence it is incident with $f_2, f_3$ and no other face of $G$. Thus $f = f_2$ or $f = f_3$ as desired.

We need the following “ear-decomposition” theorem for 2-connected graphs.

Theorem 2.3. For every 2-connected graph $G$ there exist subgraphs $G_0, G_1, \ldots, G_k$ such that

(i) $G_0$ is a cycle,

(ii) for $i = 1, 2, \ldots, k$ the graph $G_i$ is a path with both ends in $G_0 \cup G_1 \cdots \cup G_{i-1}$ and otherwise disjoint from it, and

(iii) $G = G_0 \cup G_1 \cdots \cup G_k$.

Theorem 2.4. The boundary of every face of a 2-connected plane graph is the point set of a cycle of $G$.

Proof. We proceed by induction on $|V(G)|$. If $G$ is a cycle, then the theorem follows from Theorem 1.3. By Theorem 2.3 we may therefore assume that $G$ has a 2-connected subgraph $G'$ such that $G$ is obtained from $G'$ by adding a path $P$ with ends in $G'$ and otherwise disjoint from $G'$.

Let $f \in F(G)$. Then $f \subseteq f'$ for some face $f'$ of $G'$. By induction the face $f'$ is bounded by a cycle $C$ of $G'$. If the interior of $P$ is not a subset of $f'$, then $f = f'$ by Lemma 1.10; thus $C$ is the boundary of $f$ and the theorem holds. We may therefore assume that the interior of $P$ is a subset of $f'$. Now $bd(f) \subseteq \overline{f} \cap G \subseteq C \cup P$, and hence $f$ is a face of $C \cup P$ by Lemma 1.10(ii). Thus $f$ is bounded by a cycle by Lemma 2.2.

Definition 2.5. We say that a plane graph $G$ is a straight line plane graph if each edge of $G$ is a straight line segment. We say that a straight line plane graph $G$ is a convex plane graph if every face of $G$ is bounded by a convex polygon. Thus we can speak of a straight line drawing and convex drawing of a planar graph.

Definition 2.6. We denote the graph obtained from a graph $G$ by contracting the edge $e$ by $G/e$. A minor of a graph $G$ is any graph obtained from a subgraph of $G$ by contracting edges. An $H$ minor is a minor isomorphic to $H$.

Exercise 2.7. (i) If a graph $G$ has an $H$ subdivision, then it has an $H$ minor.

(ii) If $H$ has maximum degree at most three, then a graph $G$ has an $H$ subdivision if and only it has an $H$ minor.
Exercise 2.8. Let $H$ be a fixed graph. Prove that the following statements are equivalent:

(a) For any graph $G$, the graph $G$ has an $H$ minor if and only if it has an $H$ subdivision.

(b) The graph $H$ has maximum degree at most three.

Lemma 2.9. Every 3-connected graph $G$ on at least five vertices has an edge $e$ such that $G/e$ is 3-connected.

Lemma 2.10. Every 3-connected simple graph with no $K_5$ or $K_{3,3}$ minor has a convex planar drawing.

Corollary 2.11. Every 3-connected simple planar graph has a convex planar drawing.

Our next objective is to replace minors by subdivisions in the statement of Lemma 2.10. The next lemma says that in the context of the presence of $K_5$ or $K_{3,3}$ the two are actually equivalent.

Lemma 2.12. A graph $G$ has a $K_5$ or $K_{3,3}$ minor if and only if it has a $K_5$ or $K_{3,3}$ subdivision.

Lemma 2.10 cannot be extended to all graphs, but the weaker conclusion that $G$ has a straight line drawing can. It is possible to deduce that by decomposing graphs into smaller graphs along cutsets of size at most two, but doing so carefully is not very pleasant. It seems preferable to take a different route. The following purely combinatorial lemma will allow us to bypass these technical difficulties.

Lemma 2.13. Let $G$ be a graph on at least four vertices with no $K_5$ or $K_{3,3}$ subdivision, and assume that adding an edge joining any pair of nonadjacent vertices creates a $K_5$ or $K_{3,3}$ subdivision. Then $G$ is 3-connected.

Proof. We proceed by induction on $|V(G)|$. The lemma clearly holds when $G$ has exactly four vertices, and so we may assume that $|V(G)| \geq 5$. We leave proving that $G$ is 2-connected as an exercise. Suppose for a contradiction that $G\setminus u\setminus v$ is disconnected for some two vertices $u, v \in V(G)$. Then $G$ can be written as $G_1 \cup G_2$, where $V(G_1) - V(G_2) \neq \emptyset \neq V(G_2) - V(G_1)$ and $V(G_1) \cap V(G_2) = \{u, v\}$. It follows from the maximality of $G$ that $u, v$ are adjacent in $G$, and so we may assume that they are adjacent in both $G_1$ and $G_2$. By induction each of $G_1, G_2$ is either 3-connected, or has at most three vertices. By Lemma 2.10 we may assume that $G_1, G_2$ are plane graphs. For $i \in \{1, 2\}$ let $w_i \in V(G_i)$ be such that $u, v, w_i$ belong to the boundary of some face of $G_i$.

By hypothesis the graph $G + w_1w_2$ contains a $K_5$ or $K_{3,3}$ subdivision $K$. It follows that for some $i \in \{1, 2\}$ all except possibly one branch-vertex of $K$ belong to $G_i$. From the symmetry we may assume that $i = 1$. Since $G_1$ has no $K_5$ or $K_{3,3}$ subdivision and $u, v$ are adjacent in $G_1$ we deduce that some branch-vertex $x$ of $K$ belongs to $G_2$. But $x$ is separated from the other branch-vertices of $K$ by the 3-element set $\{u, v, w_1\}$, and hence $K$ is isomorphic to $K_{3,3}$. It also follows that if in $G$ we identify all vertices of $V(G_2) - V(G_1)$ into a single vertex, the new graph is also a counterexample to the lemma. Thus it follows by induction that $G$ is equal to this graph; in other words, $G_2$ is a triangle with vertices $u, v, w_2$. But now it follows that $G + w_1w_2$ is planar, contrary to Corollary 2.1. \qed

Theorem 2.14. For a graph $G$ the following conditions are equivalent:

1. $G$ is planar,
2. $G$ has a straight line planar drawing,
3. $G$ has no $K_5$ or $K_{3,3}$ minor,
4. $G$ has no $K_5$ or $K_{3,3}$ subdivision.
Proof. Statements (3) and (4) are equivalent by Lemma 2.12. (2) \(\Rightarrow\) (1) is trivial and (1) \(\Rightarrow\) (4) follows from Corollary 2.1. To prove (3) \(\Rightarrow\) (2) let \(G\) have no \(K_5\) or \(K_{3,3}\) minor. By adding edges to \(G\) we may assume that it is edge-maximal in the sense that it satisfies the hypotheses of Lemma 2.13. By that lemma \(G\) is 3-connected, and hence it has a straight line drawing by Lemma 2.10.

Exercise 2.15. Let \(G\) be a 3-connected simple graph not isomorphic to \(K_5\). Then \(G\) is planar if and only if it has no \(K_{3,3}\) subdivision.

3 Uniqueness of planar drawings

Definition 3.1. A cycle \(C\) in a graph \(G\) is called peripheral if it is induced and \(G\setminus V(C)\) has at most one component.

Theorem 3.2. Let \(G\) be a 3-connected simple plane graph, and let \(C\) be a subgraph of \(G\). Then \(C\) bounds a face in \(G\) if and only if \(C\) is a peripheral cycle.

Proof. If \(C\) is a peripheral cycle, then by Theorem 1.3 one of the faces of \(C\), say \(f\), is disjoint from \(G\), and its boundary is \(C\). By Lemma 1.10 the face \(f\) is a face of \(G\), and hence \(C\) bounds a face of \(G\), as desired.

Conversely, let \(C\) be the boundary of a face \(f\in F(G)\). Then \(C\) is a cycle of Theorem 2.4. Suppose first that some edge \(e\in E(G) - E(C)\) has both its ends, say \(u,v\), on \(C\). Since \(G\) is simple the graph \(C\setminus u\setminus v\) consists of two components, and since \(G\) is 3-connected, there is a path \(P\) in \(G\setminus u\setminus v\) joining those two components. Let \(x,y\) be the ends of \(P\). We may assume that \(P\) is otherwise disjoint from \(C\). By adding a new vertex into the face \(f\) and joining it to \(x, y, u, v\) we obtain a planar drawing of a subdivision of \(K_5\), contrary to Lemma 2.1. This proves that \(C\) is induced.

Thus we may assume for a contradiction that \(G\setminus V(C)\) has at least two components. Let \(a,b\in V(G) - V(C)\) belong to different components of \(G\setminus V(C)\). By Menger’s theorem there exist three internally disjoint paths \(P_1, P_2, P_3\) between \(a\) and \(b\). It follows that each of \(P_1, P_2, P_3\) uses a vertex of \(C\); let \(c_i\in V(P_i)\cap V(C)\). The plane graph obtained from \(G\) by inserting a new vertex into the face \(f\) and joining it to \(c_1, c_2, c_3\) has a \(K_{3,3}\) subdivision, contrary to Lemma 2.1.