Test 2 Thursday, October 22, 2015. Details on material for which you will be responsible were sent by email this last weekend. Study hard. Experience shows that the middle portion of this course has considerably more depth and significance than the first part.
Definition  For an integer  $n \geq 1$, the poset consisting of all subsets of \{1, 2, ..., n\} ordered by inclusion is called the subset lattice. We will denote it as $2^n$. Here is $2^4$. 
**Remark** Using the alternate notation for subsets as bit strings, subset lattices are also called cubes. Here is the 4-cube.
Basic Properties of Subset Lattices

**Fact** The size of $2^n$ is $2^n$.

**Fact** The unique maximal element in $2^n$ is the set $\{1, 2, \ldots, n\}$ and the unique minimal element is the empty set $\emptyset$.

**Fact** The height of $2^n$ is $n + 1$. In fact, all maximal chains are maximum.
Basic Fact  The subset lattice $2^{n+1}$ can be viewed as $2 \times 2^n$. 
Theorem  

For \( n \geq 2 \), the \( n \)-cube subset \( 2^n \) is Hamiltonian.
Theorem  For $n \geq 2$, the $n$-cube subset $2^n$ is Hamiltonian.
The Width of Subset Lattices

**Fact** If \( A \) is a set with \( |A| = k \), then the number of maximal chains in \( 2^n \) containing \( A \) is \( k! \times (n-k)! \).

**Fact** The width of \( 2^n \) is at least as large as any binomial coefficient \( C(n, k) \), where \( 0 \leq k \leq n \).

**Fact** The largest binomial coefficient of the form \( C(n, k) \) is when \( k = \lfloor n/2 \rfloor \). When \( n \) is even, there is just one value of \( k \) for which \( C(n, k) \) is maximum. When \( n \) is odd, there are two. For example, the width of \( 2^{13} \geq C(13, 6) = C(13, 7) \) while the width of \( 2^{14} \geq C(14, 7) \).
The Width of Subset Lattices (2)

**Theorem Fact** (Sperner, '28) The width of the subset lattice \(2^n\) is the binomial coefficient \(C(n, \lfloor n/2 \rfloor)\).

**Note** We will give two proofs of this result in class. The first proof is the more classical of the two and rests on the following elementary fact.

**Fact** If \(A\) is a subset of \(\{1, 2, ..., n\}\) and \(|A| = k\), then the number of maximal chains containing \(A\) is \(k! (n - k)!\). To see this, consider bit strings. There are \(k!\) ways to add the bits in \(A\) and then another \((n - k)!\) ways to add the bits in the complement of \(A\).
Proof of Spener’s theorem  Let \( \{A_1, A_2, \ldots, A_t\} \) be a maximum antichain in \( 2^n \). For each \( i \), let \( k_i = |A_i| \). Then

\[
\sum_{1 \leq i \leq t} k_i! (n - k_i)! \leq n!
\]

\[
\sum_{1 \leq i \leq t} \left[ k_i! (n - k_i)! \right]/n! \leq 1.
\]

\[
\sum_{1 \leq i \leq t} 1/C(n, k_i) \leq 1
\]

\[
\sum_{t} 1/C(n, \lfloor n/2 \rfloor) \leq 1
\]

\[
\sum_{t} C(n, \lfloor n/2 \rfloor) \leq 1
\]
**Definition** A poset is ranked if all maximal chains are maximum. Here is one of height 4.
Definition A poset is ranked if all maximal chains are maximum. Here is one of height 5.
**Observation**  Here is the middle level for a ranked poset of height 4.
Observation  Here is the middle level for a ranked poset of height 5.
Symmetric Chain Partition

**Definition**  A chain in a ranked poset is **symmetric** when it (1) goes the same distance above and below the middle levels and (2) doesn’t skip levels.

**Observation**  Here is a symmetric chain partition for a ranked poset of height 4.
Observation  Here is a symmetric chain partition for a ranked poset of height 5.
**Lemma**  The Cartesian product of two chains has a symmetric chain partition.
Theorem. For every $n \geq 1$, the subset lattice $2^n$ has a symmetric chain partition.

Proof. On the next two slides, we illustrate the inductive construction. In each case, we start with a symmetric chain partition of $2^n$ and show how to modify two copies to obtain a symmetric chain partition of $2^{n+1}$.

Note that when $n$ is even, we have some 1-element chains in the partition. Each pair becomes a 2-element chain in the next step. But when $n$ is odd, each pair of chains produces another pair.
Symmetric Chain Partition (5)

Example  Using two copies of a symmetric chain partition of $2^2$ to form a symmetric chain partition of $2^3$. Note that the 1-element chains 2 and 23 become a 2-element chain.
Symmetric Chain Partition (6)

**Example** Using two copies of a symmetric chain partition of $2^3$ to form a symmetric chain partition of $2^4$. 