Phase Portraits of 1-D Autonomous Equations

In each of the following problems [1]-[5]: (a) find all equilibrium solutions; (b) determine whether each of the equilibrium solutions is stable, asymptotically stable or unstable; and (c) sketch the phase portrait.

[1] $\frac{dP}{dt} = P(P^2 - 1)(P - 3)$.

[2] $\frac{dy}{dt} = -(y - 1)(y - 3)^2$.

[3] $\frac{dy}{dt} = \sin(\pi y)$.

[4] $\frac{dx}{dt}(t) = \sin^2(\pi x(t))$.

[5] $\frac{dy}{dt} = f(y)$, where the function $f(y)$ is piecewise defined by:

$$f(y) = \begin{cases} 
2y & y \leq 0, \\
0 & 0 < y < 1, \\
1 - y & y \geq 1.
\end{cases}$$

[6] An equation $\frac{dy}{dt} = f(y)$ has the following phase portrait.

(a) Find all equilibrium solutions.

(b) Determine whether each of the equilibrium solutions is stable, asymptotically stable or unstable.

(c) Graph the solutions $y(t)$ vs $t$, for the initial values $y(1.4) = 0$, $y(0) = 0.5$, $y(0) = 1$, $y(0) = 1.1$, $y(0) = 1.5$, $y(-0.5) = 1.5$, $y(0) = 2$, $y(0) = 2.5$, $y(0) = 3$, $y(0) = 3.5$, $y(0) = 4$, $y(0) = 4.5$, $y(-1) = 4.5$. (Without further quantitative information about the equation and the solution formula, it’s clearly impossible to draw accurate graphs of $y(t)$ vs $t$. Here, try to sketch graphs qualitatively to show the correct dynamic properties. The point is that a great deal of info about solution dynamics can be read off from one simple figure of phase portrait.)
Several solution graphs $y(t)$ vs $t$ are given below, for an equation $\frac{dy}{dt} = f(y)$.

(a) Find all equilibrium solutions in the interval $-4 < y < 4$;
(b) Determine whether each of the above equilibrium solutions is stable, asymptotically stable or unstable;
(c) Sketch phase portrait on the interval $-4 < y < 4$.

In each of the following problems [8]-[9]:

(a) find all equilibrium solutions of the equation $(*)$;
(b) for each equilibrium point, write down the linear approximating equation near the equilibrium and determine whether the equilibrium is stable, asymptotically stable or unstable with respect to the linear approximating equation;
(c) try to use the linear stability/instability obtained in (b) to determine whether each of the equilibria is stable, asymptotically stable or unstable with respect to the nonlinear equation $(*)$;
(d) if the linear approximation obtained in (b) was not enough to determine the stability of an equilibrium with respect to the nonlinear equation $(*)$, use other methods to determine whether the equilibrium is stable, asymptotically stable or unstable with respect to the nonlinear equation $(*)$

[8] $(*) \quad \frac{dy}{dt} = y(y - 1)(y + 2)$

[9] $(*) \quad \frac{dy}{dt} = -y(y - 1)(y + 2)^2$
In each of the following problems [10]-[11]:

(a) verify that \( y = 1 \) is an equilibrium;

(b) give the linear approximating equation for \( y \approx 1 \);

(c) determine whether \( y = 1 \) is stable, asymptotically stable or unstable with respect to the nonlinear equation \((\ast)\).

[10] \((\ast)\)  \( \frac{dy}{dt} = 2y - 1 + \cos(\pi y) \).

[11] \((\ast)\)  \( \frac{dy}{dt} = 2y - 2 + \sin(\pi y) \).

(See next page for answers)
Answers:

[1] There are four equilibrium solutions: $P = -1, 0, 1, 3$. The equilibria $P = -1$ and $P = 1$ are asymptotically stable. The equilibria $P = 0$ and $P = 3$ are unstable.

[2] There are two equilibrium solutions: $y = 1, 3$. The equilibrium $y = 1$ is asymptotically stable. The equilibrium $y = 3$ is unstable.

[3] There are infinitely many equilibrium solutions: any integer is an equilibrium. Among these equilibria, odd integers $y = \pm 1, \pm 3, \pm 5, \cdots$ are asymptotically stable, while even integers $y = 0, \pm 2, \pm 4, \pm 6, \cdots$ are unstable.

[4] There are infinitely many equilibrium solutions: any integer is an equilibrium. All equilibria are unstable.

[5] There are infinitely many (actually a continuum of) equilibrium solutions: each point $y$ in the closed interval $0 \leq y \leq 1$ is an equilibrium. The equilibrium $y = 0$ is unstable. All other equilibria $0 < y \leq 1$ are stable but not asymptotically stable.
[6] There are three equilibria: $y = 1, 2, 4$. The equilibrium $y = 4$ is asymptotically stable. The equilibria $y = 1$ and $y = 2$ are unstable.

A rough sketch of the solution graphs is given below.

Besides the monotone properties and dynamic behavior of the solutions, also note that the solution graphs between $2 < y < 4$ should be all congruent. Indeed, they are horizontal translations of each other. This also holds for each of the following intervals: $1 < y < 2$, $4 < y < \infty$, and $-\infty < y < 1$.

[7] There are three equilibria: $y = -2, 0, 2$. The equilibria $y = -2$ and $y = 2$ are asymptotically stable. The equilibrium $y = 0$ is unstable.

[8] (a) There are three equilibria: $y = -2, 0, 1$.

(b) • Near $y = -2$: the linear approximating equation is (**) $\frac{dy}{dt} = 6(y + 2)$.
    The equilibrium $y = -2$ is unstable with respect to the lin approx eq (**).

• Near $y = 0$: the linear approximating equation is (**) $\frac{dy}{dt} = -2y$.
    The equilibrium $y = 0$ is asymptotically stable w.r.t. the lin approx eq (**).

• Near $y = 1$: the linear approximating equation is (**) $\frac{dy}{dt} = 3(y - 1)$.
    The equilibrium $y = 1$ is unstable with respect to the lin approx eq (**).

(c) Since each of the linear approximating equations in (b) is non-degenerate, the non-linear dynamics near the equilibrium can be qualitatively determined by the linear dynamics.
The equilibria $y = -2$ and $y = 1$ are unstable with respect to the nonlin eq (*).
The equilibrium $y = 0$ is asymptotically stable w.r.t. the nonlin eq (*).

(d) No need to consider.

[9] (a) There are three equilibria: $y = -2, 0, 1$.

(b) • Near $y = -2$: the linear approximating equation is (**) $\frac{dy}{dt} = 0$.
The equilibrium $y = -2$ is stable but not asymptotically stable, with respect to the lin approx eq (**).
• Near $y = 0$: the linear approximating equation is (**) $\frac{dy}{dt} = 4y$.
The equilibrium $y = 0$ is unstable with respect to the lin approx eq (**).
• Near $y = 1$: the linear approximating equation is (**) $\frac{dy}{dt} = -9(y - 1)$.
The equilibrium $y = 1$ is asymptotically stable w.r.t. the lin approx eq (**).

(c) • The linear approximations are sufficient to determine the nonlinear dynamics near $y = 0$ and near $y = 1$ on the qualitatively level.
The equilibrium $y = 0$ is unstable with respect to the nonlin eq (*).
The equilibrium $y = 1$ is asymptotically stable w.r.t. the nonlin eq (*).
• On the other hand, the linear approximating equation near $y = -2$ is degenerate.
The linear approximation is insufficient to determine the nonlinear dynamics near $y = -2$.

(d) For $y = -2$, the stability/instability w.r.t. the nonlinear equation (*) can be determined by studying the sign changes of the nonlinear term $f(y) = -y(y - 1)(y + 2)^2$ for $y \approx -2$.

Answer: The equilibrium $y = -2$ is unstable w.r.t. the nonlinear equation (*).

[10] (a) Let $f(y) = 2y - 1 + \cos(\pi y)$. Verify that $f(1) = 0$.

(b) $\frac{dy}{dt} = 2(y - 1)$

(c) $y = 1$ is unstable with respect to the nonlin eq (*).

[11] (a) Let $f(y) = 2y - 2 + \sin(\pi y)$. Verify that $f(1) = 0$.

(b) $\frac{dy}{dt} = (2 - \pi)(y - 1)$

(c) $y = 1$ is asymptotically stable with respect to the nonlin eq (*).