Exponential Growth/Decay

Often ① A quantity changes with time
② The rate of change is proportional to the present size of the quantity.

Equations
① \( y = y(t) \)
② \( \frac{dy}{dt} = ky \)

Initial Value Problem
\( \frac{dy}{dt} = ky \), \( y(0) = y_0 \) \( \Rightarrow \) Solution
\( y(t) = y_0 e^{kt} \)

\( k > 0 \): Exp. Growth

\( k < 0 \): Exp. Decay

\( k \) measures how fast the growth/decay is.
How to Solve $\frac{dy}{dt} = ky$?

**Method 1 (Integrating Factor)**

\[
\frac{dy}{dt} - ky = 0,
\]

Multiply both sides by $e^{-kt}$.

\[
\Rightarrow e^{-kt} \frac{dy}{dt} - ke^{-kt}y = 0.
\]

\[
\Rightarrow \frac{d}{dt}(e^{-kt}y) = 0,
\]

\[
\Rightarrow e^{-kt}y = C \Rightarrow y = Ce^{kt}
\]

**Method 2 (Separation of Variables)**

\[
\frac{dy}{dt} = ky \Rightarrow \frac{1}{y} \, dy = k \, dt
\]

\[
\Rightarrow \int \frac{1}{y} \, dy = \int k \, dt, \quad \ln |y| = kt + C_1
\]

\[
\Rightarrow |y| = e^{kt+C_1} = e^{C_1}e^{kt}
\]

\[
\Rightarrow y = \pm e^{C_1}e^{kt}
\]

\[
y = Ce^{kt}
\]
Population Growth/Decay

Linear Exp. Model (Malthus, 1798)

Assumption: Population \( y \) changes exponentially, i.e.
\[
\frac{dy}{dt} = ky, \quad y(t) = y_0 e^{kt}.
\]

Here, \( k \) = the net rate of change per unit of population
= (the birth rate) - (the death rate) per capita.

Example: A culture of bacteria grows exponentially.

Observed: # of bacteria increased 25% in a hour.

Question: How long does it take for the # of bacteria to double?

Solution: \( y(t) = y_0 e^{kt} \)

\[
y(1) = 1.25y_0 = y_0 e^{k \cdot 1} \Rightarrow 1.25 = e^k \Rightarrow k = \ln 1.25
\]

\[
y(t) = y_0 e^{(\ln 1.25)t}
\]

Q. Find \( t \) such that \( y(t) = 2y_0 \).

\[
2y_0 = y_0 e^{(\ln 1.25)t} \Rightarrow 2 = e^{(\ln 1.25)t}
\]

\[
\Rightarrow \ln 2 = (\ln 1.25)t \Rightarrow t = \frac{\ln 2}{\ln 1.25} \text{ hours}
\]
$y = y_0 e^{(\ln 1.25) t}$

**Answer:** Time to Double: \[ t_2 = \frac{\ln 2}{\ln 1.25} \approx 3.1 \text{ hours} \]

By the way:

- Time to Triple: \[ t_3 = \frac{\ln 3}{\ln 1.25} \approx 4.9 \text{ hours} \]
- Time to Quadruple: \[ t_4 = \frac{\ln 4}{\ln 1.25} \approx 6.2 \text{ hours} \]
- Time to Octuple: \[ t_8 = \frac{\ln 8}{\ln 1.25} \approx 9.3 \text{ hours} \]

We always have \[ t_4 = 2t_2, \quad t_8 = 3t_2. \]  

Q. Why?
Example

- Coffee Temp. $T(t)$
- Room Temp. 70°F (Surrounding Temp. constant)

Newton's Law of Cooling

The rate of decrease of Coffee Temp. \( \propto \) Coffee Temp. - Room Temp.

\[- \frac{dT}{dt} = k (T-70), \quad T(0)=190^\circ F.\]

[An initial value problem of diff. eq.]

Solution:

Set $y = T-70$.

\[- \frac{dy}{dt} = ky.\]  \[\begin{cases} \frac{dy}{dt} = -ky \\ y(0) = 190-70 = 120 \end{cases} \]

Solution: $y(t) = 120 e^{-kt}$

Coffee Temp. $T(t) = 70 + y$

$= 70 + 120 e^{-kt}$
Example: A Hot Egg in Water.

Egg Temp. = T : changes with t
Water Temp. = Constant = T_s
(Surrounding Temp.)

Assume:
- Initial Egg Temp. \( T_0 = 98^\circ C \) \((t=0)\)
- Constant Water Temp. \( T_s = 18^\circ C \).
- Egg Temp. after 5 min: \( T(5) = 38^\circ C \).

Q. Find \( t \) such that \( T(t) = 20^\circ C \).

Solution: Newton's Cooling Law: \( \frac{dT}{dt} = -k(T-T_s) \)

Set: \( y = T-T_s \) \( \Rightarrow \frac{dy}{dt} = -ky \)

\( \Rightarrow y = y_0 e^{-kt} \)

\( T-T_s = (T_0-T_s) e^{-kt} \), \( T-T_s = (98-18)e^{-kt} \)

\( T(t) = 18+80 e^{-kt} \)

\( t=5: 38-18 = 80 e^{-5k} \) \( \Rightarrow e^{5k} = \frac{20}{80} = \frac{1}{4} \),

\( -5k = \ln\left(\frac{1}{4}\right) = -\ln 4 \), \( k = \frac{1}{5}\ln 4 \).

Eq: \( 20-18 = 80 e^{\left(-\frac{1}{5}\ln 4\right) t} \), \( \left(-\frac{1}{5}\ln 4\right)t = \ln\frac{3}{80} \) \( \Rightarrow t = \frac{5\ln 40}{\ln 4} \).
\[ T(t) = 18 + 80 e^{-kt} \]

\[ T(0) = 98, \quad T(5) = 38 \Rightarrow k = \frac{1}{5} \ln 4 \]

\[ T(t) = 18 + 80 e^{-\left(\frac{1}{5} \ln 4\right)t}, \quad T(t) = 25 \Rightarrow t = ? \]

**Ans:** \[ t = \frac{5 \ln 40}{\ln 4} \text{ min} \approx 13.30 \text{ min} \approx 13 \text{ min } 18 \text{ sec.} \]
Radioactive Decay

Physics $\Rightarrow$ Radioactive Matters

Decay Exponentially.

$y =$ the amount of radioactive

\[
\frac{dy}{dt} = -ky \quad (k > 0: \text{constant}) \quad y(t) = y_0 e^{-kt}
\]

- $k$: the decay rate (measures how fast the decay is).
- Half-Life: also measures how fast the decay is.

Original Sample $\xrightarrow{\text{decay}}$ Half of the original size

Half-Life $h =$ the time length required for decaying into half.

\[
y(h) = y_0 e^{-kh} = \frac{1}{2} y_0 \Rightarrow e^{-kh} = \frac{1}{2}
\]

\[-kh = \ln \frac{1}{2} = -\ln 2
\]

$kh = \ln 2$
Carbon Dating

C-12 (6 neutrons) : stable carbon
C-14 (8 neutrons) : radioactive isotope

Facts from Physics

- In a living organism:
  \[
  \frac{\text{Amount of C-14}}{\text{Amount of C-12}} = \frac{\text{C-14 in Atmosphere}}{\text{C-12 in Atmosphere}} = \text{Const.} = 10^{-12}
  \]

- When an organism dies:
  \[
  \begin{align*}
  \text{C-14} : & \quad \text{decays exponentially with half-life 5730 years} \\
  \text{C-12} : & \quad \text{NO change}
  \end{align*}
  \]

Method of Carbon Dating

In an ancient (prehistoric) artifact, measure the levels of C-14 & C-12.

⇒ the age of the artifact
Example. The Shroud of Turin.

Some people believed that it was used to bury Jesus Christ.

Example (Shroud of Turin)

Tested in 1988:

(Amount of C-14) = 92.27% of the original size

⇒ Age of Shroud = ?

Solution (half-life of C-14) = 5730

Amount of C-14: \( y(t) = y_0 e^{-kt} \)

\[ k \cdot 5730 = \ln 2, \quad k = \frac{\ln 2}{5730}. \]

\[ y(t) = y_0 e^{-\left(\frac{\ln 2}{5730}\right)t} \]

\[ y_0 e^{-\left(\frac{\ln 2}{5730}\right)t} = 0.9227 y_0 \]

⇒ \(-\left(\frac{\ln 2}{5730}\right)t = \ln 0.9227 \) age

\[ t = -\frac{\ln 0.9227}{\ln 2} \cdot 5730 \approx 665 \text{ years} \]

⇒ \( 1988 - 665 = 1323 \).

Conclusion: The shroud was made around the year of 1323.