Scalar Separable Equations: \( \frac{dy}{dx} = f(x)g(y) \)

Solution Method: Rewrite the given equation (symbolically):
\[
\frac{dy}{g(y)} = f(x)dx.
\]

Now integrate both sides:
\[
\int \frac{dy}{g(y)} = \int f(x)dx.
\]
This gives a relation between \( x \) and \( y \). If you can solve this equation for \( y \) in terms of \( x \), go ahead.

Example: Solve the initial value problem \( \frac{dy}{dx} = e^{2y} \cos x, y(0) = -\frac{1}{3} \).

Solution: Rewrite the diff. eq. into \( e^{-2y}dy = \cos xdx \) and then integrate:
\[
\int e^{-2y}dy = \int \cos xdx.
\]
This gives
\[
-\frac{1}{2}e^{-2y} = \sin x + C,
\]
where \( C \) is an arbitrary constant. Solve the last equation for \( y \) in terms of \( x \):
\[
y = -\frac{1}{2} \ln(-2 \sin x - 2C).
\]
This is the general solutions to the given ODE. Now examine the initial condition:
\[
y(0) = -\frac{1}{3} \quad \Rightarrow \quad -\frac{1}{3} = -\frac{1}{2} \ln(-2C) \quad \Rightarrow \quad C = -\frac{1}{2}e^{2/3}.
\]
Thus, the answer to the problem is
\[
y = -\frac{1}{2} \ln(-2 \sin x + e^{2/3}).
\]
Exercises

Solve the following ODEs and initial value problems of ODEs.

[1] \( y'(t) + (2t - \sin 2t)y(t) = 0 \)

[2] \( tx'(t) + 4x(t) = 0 \) \( (t > 0) \), \( x(3) = 2 \)

[3] \( y'(x) = x^2 e^x y(x)^2 \)

[4] \( \frac{dy}{dx} = (1 + 6x)y(1 - y) \), \( y(0) = 1/3 \)

Answers

[1] \( y = Ce^{-t^2 - 0.5 \cos 2t} \)

[2] \( x(t) = 162t^{-4} \)

[3] \( y = -1/(C + x^2 e^x - 2xe^x + 2e^x) \)

[4] \( y = \frac{e^{x+3x^2}}{2 + e^{x+3x^2}} \)