Second Order Homogeneous Linear Differential Equations:
the method of reduction of order

Xu-Yan Chen
Second Order Homogeneous Linear Differential Equations:

\[ a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0 \]

General solution structure:

\[ y(t) = C_1y_1(t) + C_2y_2(t) \]

where \( y_1(t) \) and \( y_2(t) \) are two linearly independent solutions.

No general solution method.
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If you can give me one, just one, nonzero solution \( y_1(t) \),
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No general solution method.

What is this note about? Reduction of Order.

If you can give me one, just one, nonzero solution \( y_1(t) \), I will get you all solutions.
Second Order Homogeneous Linear Differential Equations:

\[(\ast)y \quad a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0.\]

The Method of Reduction of Order:
Second Order Homogeneous Linear Differential Equations:

\[(\ast)y \quad a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0.\]

The Method of Reduction of Order:

- To start, a solution \(y_1(t) \neq 0\) needs to be provided/prepared.
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\[ (*) \ y \quad a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0. \]

The Method of Reduction of Order:

- To start, a solution \( y_1(t) \neq 0 \) needs to be provided/prepared.
- Set \( y(t) = y_1(t)u(t) \). 

Substitute \( y(t) = y_1(t)u(t) \) in the eq \( (*) \), it simplifies to

\[ (*) \ u \quad b_2(t)u''(t) + b_1(t)u'(t) = 0. \]

Set \( v(t) = u'(t) \). The eq \( (*) \) becomes a 1st order linear eq:

\[ (*) \ v \quad b_2(t)v''(t) + b_1(t)v'(t) = 0. \]

Solve \( (*) \) \( v \) by either integrating factor or separating the variables (both work).

Get \( u(t) \) from \( v(t) \):

\[ u(t) = \int v(t) \, dt. \]

Finally, get \( y(t) \) from \( u(t) \):

\[ y(t) = y_1(t)u(t). \]
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- Set \(y(t) = y_1(t)u(t)\).
- Substitute \(y(t) = y_1(t)u(t)\) in the eq \((*)_y\). It simplifies to
  \[(*)_u \quad b_2(t)u''(t) + b_1(t)u'(t) = 0.\]
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Details of the derivation:

\[a_2(t)(y_1u)'' + a_1(t)(y_1u)' + a_0(t)y_1u = 0,\]
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\begin{align*}
  a_2(t)(y_1 u)'' + a_1(t)(y_1 u)' + a_0(t)y_1 u &= 0, \\
  a_2(y_1 u'' + 2y_1' u' + y_1'' u) + a_1(y_1 u' + y_1' u) + a_0 y_1 u &= 0,
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\[a_2(t)(y_1u)'' + a_1(t)(y_1u)'+ a_0(t)y_1 u = 0,\]
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- Set \(v(t) = u'(t)\). The eq \((*)_u\) becomes a 1st order linear eq:
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  \[ (*)_v \quad b_2(t)v'(t) + b_1(t)v(t) = 0. \]
- Solve \( (*)_v \) by either *integrating factor* or *separating the variables* (both work).
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- Get \(u\) from \(v\): \(u(t) = \int v(t)dt\).
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- Solve \((\ast)_v\) by either integrating factor or separating the variables (both work).
- Get \(u\) from \(v\): \(u(t) = \int v(t)dt\).
- Finally, get \(y\) from \(u\): \(y(t) = y_1(t)u(t)\).
Example 1: Find general solutions of

\[(*)_y \quad t^2(1 + 2t)y'' - 2t(1 + t)y' + 2(1 + t)y = 0,\]

by using the fact that \(y_1(t) = t\) is a particular solution.
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- Set \(y(t) = y_1(t)u(t) = tu(t)\) and substitute \(y(t) = tu(t)\) in \((*)_t\):

\[t^2(1 + 2t)(tu)'' - 2t(1 + t)(tu)' + 2(1 + t)tu = 0,\]
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t^2(1 + 2t)(tu)'' - 2t(1 + t)(tu)' + 2(1 + t)tu = 0,
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t^2(1 + 2t)(tu'' + 2u') - 2t(1 + t)(tu' + u) + 2(1 + t)tu = 0,
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\text{Set } y(t) &= y_1(t)u(t) = tu(t) \text{ and substitute } y(t) = tu(t) \text{ in } (\ast)_y:
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\quad t^3(1 + 2t)u'' + 2t^3u' &= 0, \\
\quad (\ast)_u \quad (1 + 2t)u'' + 2u' &= 0.
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- Eq \((\ast)_v\) can be solved by either integrating factor or separating variables (both work). Solve \((\ast)_v \Rightarrow v(t) = \frac{C_1}{1 + 2t}.\)

Or, equivalently, \(y(t) = C_1 t^2 \ln(1 + 2t) + C_2 t\).
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- Get \(u\) from \(v\):
  \[u' = v \quad \Rightarrow \quad u(t) = \int v(t)dt = \int \frac{C_1}{1 + 2t}dt = C_1 \frac{1}{2} \ln(1 + 2t) + C_2\]
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- Finally, get \(y\) from \(u\):
  \[y(t) = y_1(t)u(t) = tu(t) = t \left[ C_1 \frac{1}{2} \ln(1 + 2t) + C_2 \right].\]
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Or, equivalently, \(y(t) = C_1 \frac{t}{2} \ln(1+2t) + C_2t\).
**Example 2:** Find general solutions of

\[(*)_y \quad 4y'' - 12y' + 9y = 0.\]
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\[(*) \quad 4y'' - 12y' + 9y = 0.\]

- Characteristic polynomial $4\lambda^2 - 12\lambda + 9 = 0$
  $\Rightarrow$ Repeated Characteristic roots: $\lambda_1 = \lambda_2 = 3/2$
  $\Rightarrow$ A particular solution $y_1(t) = e^{3/2 t}$. 
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- Set \( y(t) = y_1(t)u(t) = e^{3t/2}u(t) \) and substitute this in \( (*)_y \):
  \[ 4(e^{3t/2}u)'' - 12(e^{3t/2}u)' + 9e^{3t/2}u = 0, \]
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  \[4(e^{\frac{3}{2}t}u)'' - 12(e^{\frac{3}{2}t}u)' + 9e^{\frac{3}{2}t}u = 0,\]
  
  \[4(e^{\frac{3}{2}t}u'' + 3e^{\frac{3}{2}t}u' + \frac{9}{4}e^{\frac{3}{2}t}u) - 12(e^{\frac{3}{2}t}u' + \frac{3}{2}e^{\frac{3}{2}t}u) + 9e^{\frac{3}{2}t}u = 0,\]
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  4e^{3t/2}u'' = 0,
  
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- Characteristic polynomial \(4\lambda^2 - 12\lambda + 9 = 0\)
  \(\Rightarrow\) Repeated Characteristic roots: \(\lambda_1 = \lambda_2 = 3/2\)
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- Set \(y(t) = y_1(t)u(t) = e^{3/2t}u(t)\) and substitute this in \((*)_y\):
  \[4(e^{3/2t}u)'' - 12(e^{3/2t}u)' + 9e^{3/2t}u = 0,\]
  \[4(e^{3/2t}u'' + 3e^{3/2t}u' + \frac{9}{4}e^{3/2t}u) - 12(e^{3/2t}u' + \frac{3}{2}e^{3/2t}u) + 9e^{3/2t}u = 0,\]
  \[4e^{3/2t}u'' = 0,\]
  \((*)_u\) \quad \(u'' = 0\).
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- Set \(y(t) = y_1(t)u(t) = e^{3/2t}u(t)\) and substitute this in \((\ast)_y\):
  
  \[
  4(e^{3/2}u)'' - 12(e^{3/2}u)' + 9e^{3/2}u = 0, \\
  4(e^{3/2}u)' + 3e^{3/2}u' + \frac{9}{4}e^{3/2}u - 12(e^{3/2}u' + \frac{3}{2}e^{3/2}u) + 9e^{3/2}u = 0, \\
  4e^{3/2}u'' = 0, \\
  (\ast)_u \quad u'' = 0.
  \]

- Integrate once: \(u' = C_1\)
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- Set \(y(t) = y_1(t)u(t) = e^{3/2t}u(t)\) and substitute this in \((*)_y:\)
  \[4(e^{3/2t}u)'' - 12(e^{3/2t}u)' + 9e^{3/2t}u = 0,\]
  \[4(e^{3/2t}u'' + 3e^{3/2t}u' + \frac{9}{4}e^{3/2t}u) - 12(e^{3/2t}u' + \frac{3}{2}e^{3/2t}u) + 9e^{3/2t}u = 0,\]
  \[4e^{3/2t}u'' = 0,\]
  \((*)_u \quad u'' = 0.\)

- Integrate once: \(u' = C_1\)

- Integrate again: \(u(t) = C_1t + C_2\)
Example 2: Find general solutions of

\[(\ast) y \quad 4y'' - 12y' + 9y = 0.\]

- Characteristic polynomial \(4\lambda^2 - 12\lambda + 9 = 0\)
  \(\Rightarrow\) Repeated Characteristic roots: \(\lambda_1 = \lambda_2 = \frac{3}{2}\)
  \(\Rightarrow\) A particular solution \(y_1(t) = e^{\frac{3}{2}t}\).

- Set \(y(t) = y_1(t)u(t) = e^{\frac{3}{2}t}u(t)\) and substitute this in \((\ast)y\):
  \[4(e^{\frac{3}{2}t}u)'' - 12(e^{\frac{3}{2}t}u)' + 9e^{\frac{3}{2}t}u = 0,\]
  \[4(e^{\frac{3}{2}t}u'' + \frac{3}{2}e^{\frac{3}{2}t}u' + \frac{9}{4}e^{\frac{3}{2}t}u) - 12(e^{\frac{3}{2}t}u' + \frac{3}{2}e^{\frac{3}{2}t}u) + 9e^{\frac{3}{2}t}u = 0,\]
  \[4e^{\frac{3}{2}t}u'' = 0,\]
  \[(\ast)u \quad u'' = 0.\]

- Integrate once: \(u' = C_1\)

- Integrate again: \(u(t) = C_1t + C_2\)

- Finally, get \(y\) from \(u\): \(y(t) = y_1(t)u(t) = e^{\frac{3}{2}t}u(t)\).
  \[y(t) = C_1 te^{\frac{3}{2}t} + C_2 e^{\frac{3}{2}t}.\]